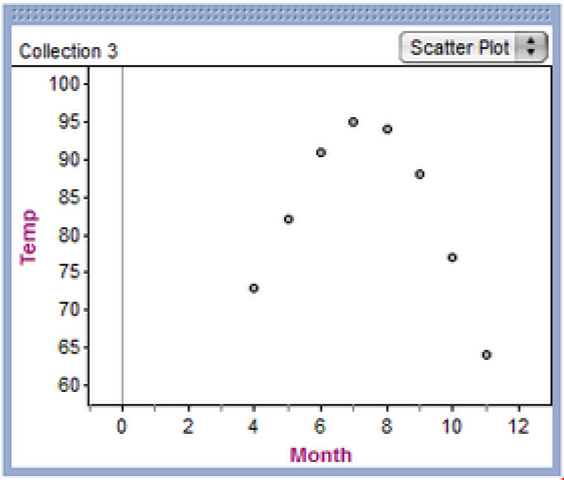
4 – Circular Functions

Problem set 4-1

Problems 1 and 2 are a quick introduction to the chapter. Let's do these in class now and you can complete the rest of the problem set for homework.

1. Recall this data set from problem 2-7-1:
The table and graph below give the average monthly temperatures (F) for Phoenix, AZ.



Collection 3

	Month	Temp
1	4	73
2	5	82
3	6	91
4	7	95
5	8	94
6	9	88
7	10	77
8	11	64

Source: *Phoenix Average Monthly Temperatures*. About.com, n.d. Web. 25 Feb. 2013. <http://phoenix.about.com/od/weather/a/averagetemps.htm>.

Answer all parts of the question in the context of the Phoenix weather data.

a) What is the interpretation of the ordered pair (7, 95)?
b) If the data set were to be extended, what would month 19 represent?
c) What would you expect the average monthly temperature to be in month 19?
d) Describe the general pattern of the data over several years. What type of model would be appropriate for this data?
e) What property of this data over the years makes it different than any data set we could model in Chapter 2?

2. Name three phenomena that have values that increase and decrease repeatedly over time.

3. Go to www.google.com/trends and find a word (not previously discussed) that has been Googled periodically and find the period. Then find a word that has *not* been Googled in any periodic pattern.

4. Convert 1 degree into radians. Convert 1 radian into degrees.

5. Convert 3.14 radians into degrees.

6. Answer the following questions about radians:
   a) What is a radian? Is it a length? Explain.
   b) Why are radians called radians?

7. Fill in the missing angle measures in both degrees and radians. An example is given. Don't forget the quadrantal angles!

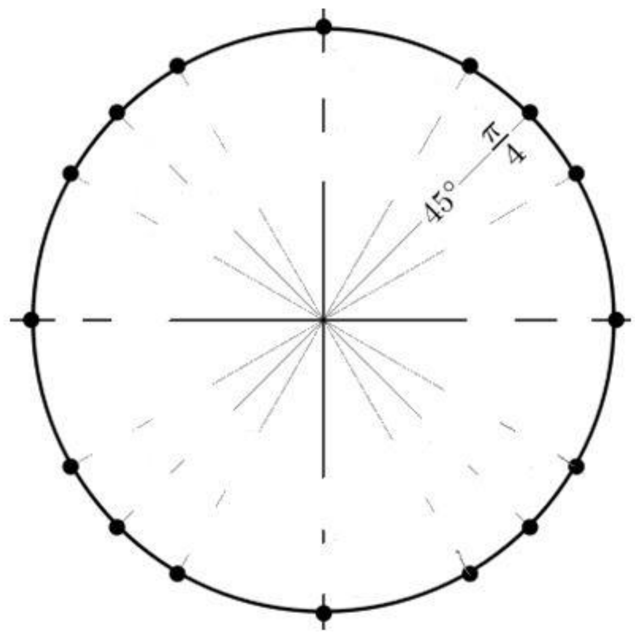


Image Source: *The Python Game Book*. Horst Jens, etc., n.d. Web. 25 Feb. 2013. <http://thepythongamebook.com/en:pygame:step017>.

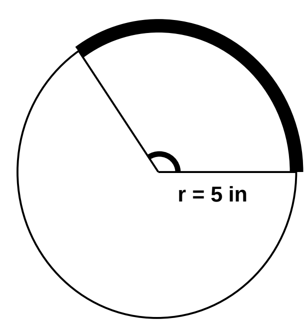
8. Convert -270° into revolutions.

9. Convert 1 counter-clockwise revolution into radians.

10. The unit circle is defined as a circle centered at the origin with a radius of 1. What is true about the radian measure of any angle in a unit circle?

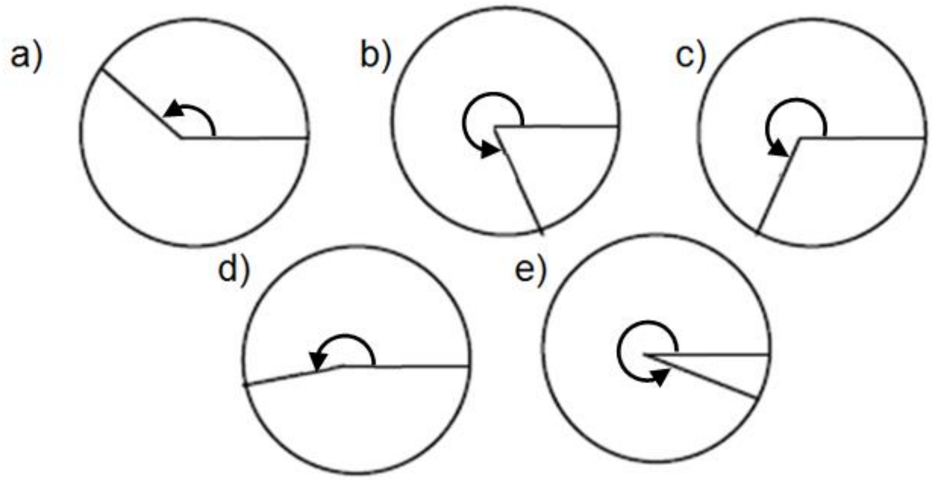
11. Consider the circle at right.
    a) Using the radius as a scale, estimate the length of the arc in bold.

    b) Using your estimate from part a, give the measure of the marked angle in radians.



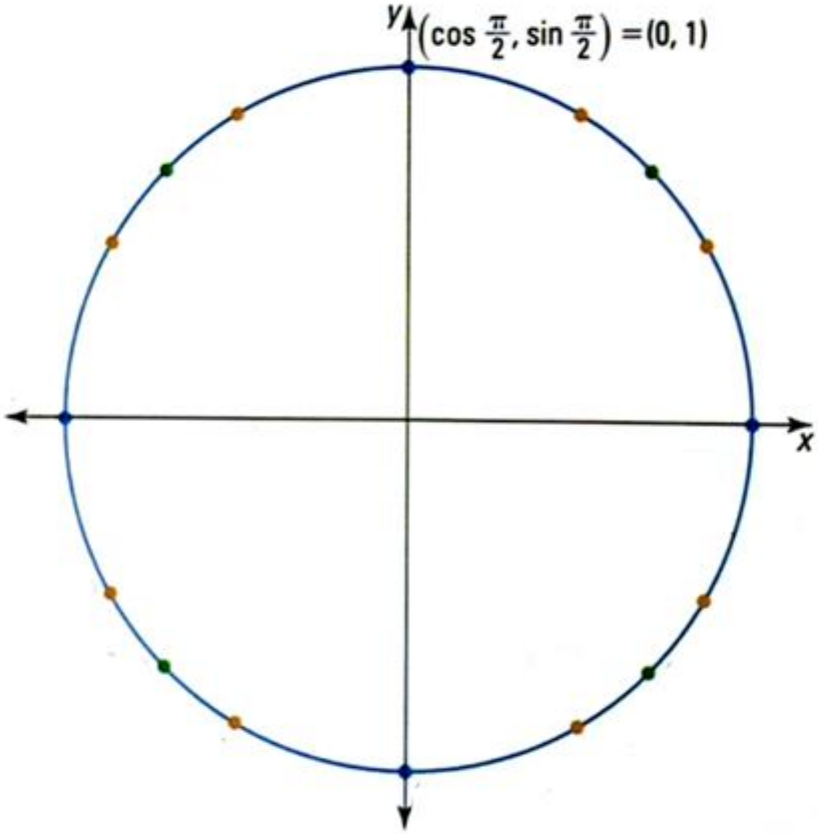
12. Note that angles measured in degrees are written as numbers followed by a degree symbol, but angles measured in radians have no symbol after the number (for example: $\frac{\pi}{2}$), indicating that radians have no units. Why is it that radians have no units or symbol?

13. Estimate the radian measure of each angle shown:



Problem Set 4-2

1. The figure below shows coordinates of the point (1, 0) after a rotation of $\frac{\pi}{2}$ radians about the origin. The coordinates are given both in terms of cos and sin as well as numbers. Fill in the missing coordinates in the same way for the point (1, 0) after a rotation of 0, $\frac{\pi}{4}$, π, and $\frac{3\pi}{2}$.



2. Refer to your answers in problem 1 to help you evaluate the following expressions for cos and sin. Use those values and the definition of tangent to evaluate the expressions for tangent.
   a) $\cos(0)$ $\sin(0)$ $\tan(0)$
   b) $\cos\left(\frac{\pi}{4}\right)$ $\sin\left(\frac{\pi}{4}\right)$ $\tan\left(\frac{\pi}{4}\right)$
   c) $\cos\left(\frac{\pi}{2}\right)$ $\sin\left(\frac{\pi}{2}\right)$ $\tan\left(\frac{\pi}{2}\right)$
   d) $\cos(\pi)$ $\sin(\pi)$ $\tan(\pi)$
   e) $\cos\left(\frac{3\pi}{2}\right)$ $\sin\left(\frac{3\pi}{2}\right)$ $\tan\left(\frac{3\pi}{2}\right)$

3. Refer to your answers in problem 1 to help you answer the following questions:
   a) Why is sin(0) = sin(π) ?

   b) Which of the angles for which you found coordinates have the same cosine value and why?

   c) What range of angles between 0 and 2π have a positive sine value and why?

   d) What range of angles between 0 and 2π have a negative cosine value and why?

   e) Using the values of $\cos(0)$ and $\sin(0)$ and the definition of the tangent function, calculate $\tan(0)$. Do the same for $\tan\left(\frac{\pi}{2}\right)$, $\tan(\pi)$ and $\tan\left(\frac{3\pi}{2}\right)$.

   f) The coordinates at the angle $\frac{3\pi}{4}$ are not labeled in the diagram in (1), but use the coordinates at $\frac{\pi}{4}$ to make a conjecture about the coordinates of the point at $\frac{3\pi}{4}$.

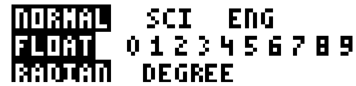
4. How can you express the exact coordinates of the image of (1, 0) after a rotation of 87° about the origin?

5. Estimate the coordinates of the image of (1,0) after a rotation of 87° about the origin by drawing a picture of the unit circle. Check to see how close your estimate is with a calculator by evaluating your answer in problem #4 (1).

6. Draw a picture using the unit circle to explain why $\cos\left(\frac{\pi}{2}\right) = 0$.

Problem Set 4-3

You can evaluate trigonometric expressions of special angles, but if the answer is irrational, a graphics calculator without CAS (Computer Algebra System) will give only an approximation.



For example, make sure you are in “radian mode” and evaluate $\sin\left(\frac{2\pi}{3}\right)$. The result

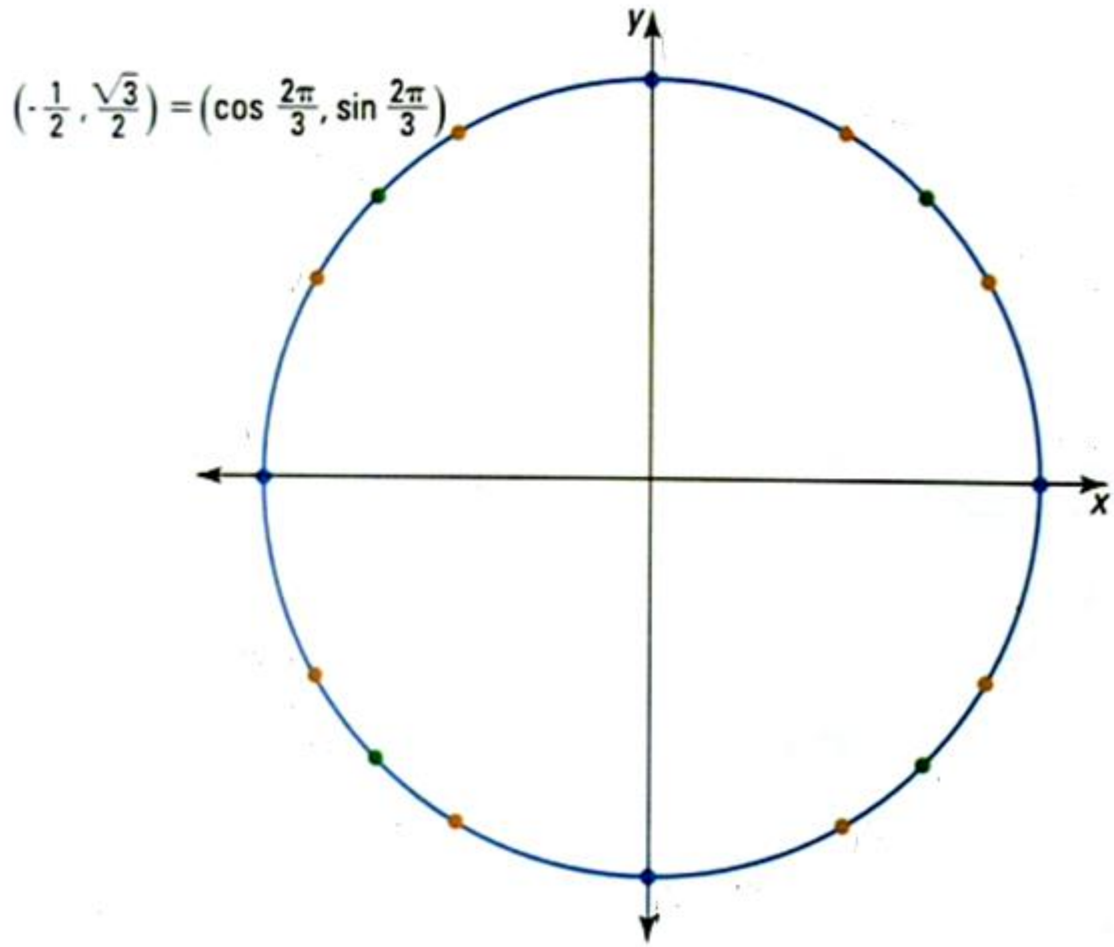
is

Note, the value given is an *approximation* of $\sin\left(\frac{2\pi}{3}\right)$ and not an *exact value*.

Wolfram Alpha will give you both the approximation and exact value as shown below. Because Computer Algebra Systems (like Wolfram Alpha) are available for free on the web, this unit has a decreased emphasis on knowing, for example, that $\sin\left(\frac{2\pi}{3}\right)=\frac{\sqrt{3}}{2}$ and an increased emphasis on explaining *why* $\sin\left(\frac{2\pi}{3}\right)=\frac{\sqrt{3}}{2}$.

WolframAlpha. Wolfram Research, n.d. Web. 26 Apr. 2013.
http://www.wolframalpha.com/input/?i=sin%282+pi%2F3%29.

1. The figure below has coordinates filled in for the point (1, 0) after a rotation of $\frac{2\pi}{3}$ radians about the origin. The coordinates are given both in terms of cos and sin as well as numbers. Fill in the missing coordinates in the same way for all "special angles". This may look like an onerous task, but there is a pattern to the values you're looking for which makes the task much easier. Make sure you understand the pattern instead of treating each coordinate as a separate problem.



2. Refer to your answer to problem 1 to help you answer the following questions:
   a) What are the possible values for the sine of any special angle?

   b) What are the possible values for the cosine of any special angle?

   c) Are the sine and cosine of a given angle ever the same between 0 and 2π? If so, where and why?

   d) Why is $\sin\left(\frac{\pi}{6}\right) = \sin\left(\frac{5\pi}{6}\right)$?

   e) Using the values of $\cos\left(\frac{\pi}{4}\right)$ and $\sin\left(\frac{\pi}{4}\right)$ and the definition of tangent, calculate $\tan\left(\frac{\pi}{4}\right)$. Do the same for $\tan\left(\frac{3\pi}{4}\right)$, $\tan\left(\frac{5\pi}{4}\right)$ and $\tan\left(\frac{7\pi}{4}\right)$.

3. Refer to your answer to problem 1 to help you evaluate the following expressions:

a) $\sin\left(\frac{\pi}{3}\right)$

b) $\cos\left(\frac{4\pi}{3}\right)$

c) $\sin\left(\frac{11\pi}{3}\right)$

d) $\cos\left(\frac{5\pi}{6}\right)$

e) $\cos\left(-\frac{2\pi}{3}\right)$ $\quad \sin\left(-\frac{2\pi}{3}\right)$ $\quad \tan\left(-\frac{2\pi}{3}\right)$

f) $\cos(180^\circ)$ $\quad \sin(180^\circ)$ $\quad \tan(180^\circ)$

g) $\cos(-135^\circ)$ $\quad \sin(-135^\circ)$ $\quad \tan(-135^\circ)$

h) $\cos(300^\circ)$ $\quad \sin(300^\circ)$ $\quad \tan(300^\circ)$

4. Draw a picture that includes the unit circle to explain why…

a) $\sin(135^0) = \frac{\sqrt{2}}{2}$

b) $\cos\left(\frac{7\pi}{6}\right) = -\frac{\sqrt{3}}{2}$

Problem Set 4-4

You need to be able to do 1 and 2 neatly, accurately, and without a calculator. Practice so that you can draw each graph in about 1 minute.

1. Sketch $y = \sin(x)$ for $-2\pi \le x \le 2\pi$.

2. Sketch $y = \cos(x)$ for $-2\pi \le x \le 2\pi$.

Recall from Chapter 3 that we graphed $y = x^2$ and $y = (x-3)^2$ and determined what the effect was of replacing x with x-3. Similarly, we graphed $y = x^2$ and $y - 1 = x^2$ and determined what the effect was of replacing y with y-1. We called the generalization of that principle the Graph Translation Theorem. Use that principle to sketch the graphs in (3) and (4). Sketch the graphs without your calculator and then confirm you have the correct answer with your calculator.

3. Sketch $y - 1 = \sin(x)$, which is the same as $y = \sin(x) + 1$, for $-2\pi \le x \le 2\pi$.
4. Sketch $y = \cos\left(x - \frac{\pi}{2}\right)$, for $-2\pi \le x \le 2\pi$.

Also, recall from Chapter 3 that we graphed $y = x^2$ and $\frac{y}{3} = x^2$ and determined what the effect was of replacing y with $\frac{y}{3}$. We called the generalization of that principle the Graph Scale Change Theorem. Use that principle to sketch the graphs in 5 and 6.

5. Sketch $\frac{y}{3} = \sin(x)$, which is the same as $y = 3\sin(x)$, for $-2\pi \le x \le 2\pi$.

6. Sketch $y = \sin\left(\frac{x}{2}\right)$, for $-2\pi \le x \le 2\pi$.

7. For the function $f(x) = \sin(x)$:
   a) What is the domain?
   b) What is the range?
   c) For what values of *x* in the range $-2\pi \le x \le 2\pi$ is the function increasing?

8. For the function $f(x) = \cos(x)$:
   a) What is the domain?
   b) What is the range?
   c) For what values of x in the range $-2\pi \le x \le 2\pi$ is the function decreasing?

9. What is the largest output value (value of y) of $y = \sin(x) - 3$? Explain how you got your answer.

10. What is the largest output value (value of y) of $y = 4\cos(x)$? Explain how you got your answer.

Problem Set 4-5

1. What is the definition of $\tan(x)$?
2. Why is $\tan\left(\frac{\pi}{4}\right) = 1$?
3. The following is a review question from Problem Set 4-3.
   Evaluate the following expressions:

   a) $\cos(0)$ $\sin(0)$ $\tan(0)$
   b) $\cos\left(\frac{\pi}{4}\right)$ $\sin\left(\frac{\pi}{4}\right)$ $\tan\left(\frac{\pi}{4}\right)$
   c) $\cos\left(\frac{\pi}{2}\right)$ $\sin\left(\frac{\pi}{2}\right)$ $\tan\left(\frac{\pi}{2}\right)$
   d) $\cos\left(\frac{3\pi}{4}\right)$ $\sin\left(\frac{3\pi}{4}\right)$ $\tan\left(\frac{3\pi}{4}\right)$
   e) $\cos(\pi)$ $\sin(\pi)$ $\tan(\pi)$
   f) $\cos\left(\frac{5\pi}{4}\right)$ $\sin\left(\frac{5\pi}{4}\right)$ $\tan\left(\frac{5\pi}{4}\right)$
   g) $\cos\left(\frac{3\pi}{2}\right)$ $\sin\left(\frac{3\pi}{2}\right)$ $\tan\left(\frac{3\pi}{2}\right)$
   h) $\cos\left(\frac{7\pi}{4}\right)$ $\sin\left(\frac{7\pi}{4}\right)$ $\tan\left(\frac{7\pi}{4}\right)$
   i) $\cos(2\pi)$ $\sin(2\pi)$ $\tan(2\pi)$

4. For what values of x on the interval $-2\pi \le x \le 2\pi$ is $\tan(x)$ undefined? Explain why.
5. Sketch $y = \tan(x)$ for $-2\pi \le x \le 2\pi$.
6. Looking at your graph from problem 5, you can see that the tangent function is periodic.
   a) What is the period of the tangent function?
      Is it the same as the period of sine and cosine?
   b) Show that $\tan\left(\frac{\pi}{4}\right) = \tan\left(\frac{\pi}{4} + \pi\right)$.
7. Find the exact value of $\tan(100.5\pi)$.
8. For the function $f(x) = \tan(x)$
   a) What is the domain?
   b) What is the range?

Problem Set 4-6

This problem set will probably take two days.

1. Learning the theory – Either in class or for homework, view two different resources that will explain how to model with the function $y = \sin(x)$. Note: the theory is explained using y = sin(x) as the modeling function, but y = cos(x) could just as easily be used; it is a matter of personal preference.
   a) Modeling with y=sin(x) (FluidMath video)
   b) Modeling with y=sin(x) (PPT)

2. Practice the theory – There are four resources that will help you practice the theory. The first two are PowerPoint files
   a) Modeling with y=sin(x) and y=cos(x) - **exact** model practice (PPT)
   b) Modeling with y=sin(x) - **approximate** model practice (PPT)
   Note: This second PowerPoint uses y = sin(x) as the modeling function, but y = cos(x) could just as easily be used.

   The last two are Fathom files
   c) Modeling with y=**sin**(x) - approximate model practice (Fathom)
   d) Modeling with y=**cos**(x) - approximate model practice (Fathom)

In 3 – 5, complete parts a – e and then any additional parts of that problem.
Note: you can choose to use either $y = \sin(x)$ or $y = \cos(x)$ as the parent function.

a) Find an equation in implicit form $\frac{y-k}{a} = \sin\left(\frac{x-h}{b}\right)$ that models the data.

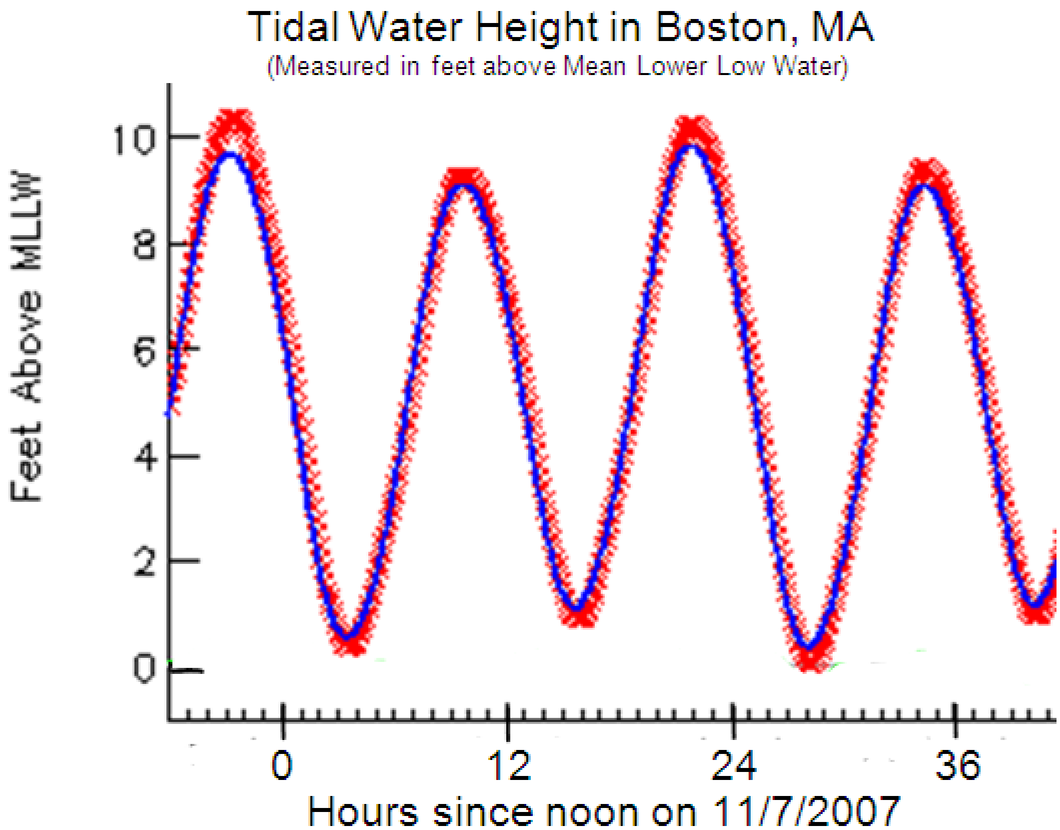
b) Rewrite the equation in function form $y = a \cdot \sin\left(\frac{x-h}{b}\right) + k$.

c) Enter your equation from part c into your calculator and graph it on the same window as the graph shown in the problem. They should look approximately the same.

d) Give the period and amplitude for your model and explain what they mean within the context of the problem.

e) Give the coordinates of the point (h, k) for your model and explain what they mean within the context of the problem.

3. For the tidal water graph below, do parts a – e as outlined above and then answer two more questions:
   f) Use the graph to estimate how many feet above Mean Lower Low Water (MLLW) the tide was 24 hours after noon on 11/7/2007.
   g) Use your equation from part b to calculate how many feet above MLLW the tide was 24 hours after noon on 11/7/2007.



Source: *Water Level Stations by State*. NOAA, n.d. Web. 7 Nov. 2007. <http://tidesonline.nos.noaa.gov/ geographic.html>.

4. For the average daily low temperature in Concord, NC graph below (the blue curve), do parts a – e as outlined above and then answer three more questions:

   f) Martin Luther King, Jr. Day is January 20^{th}, the 20^{th} day of the year. Use the graph to estimate the mean low temperature in Concord, NC on MLK Day.

   g) Use your equation from part b to calculate the mean low temperature in Concord, NC on MLK Day.

   h) Recall that a function is periodic with period P if $f(x+P)=f(x)$ for all values of x. Show that your model is periodic by demonstrating that $f(x+P)=f(x)$ for January 15^{th} (x=15). Do this with the equation, not the graph.

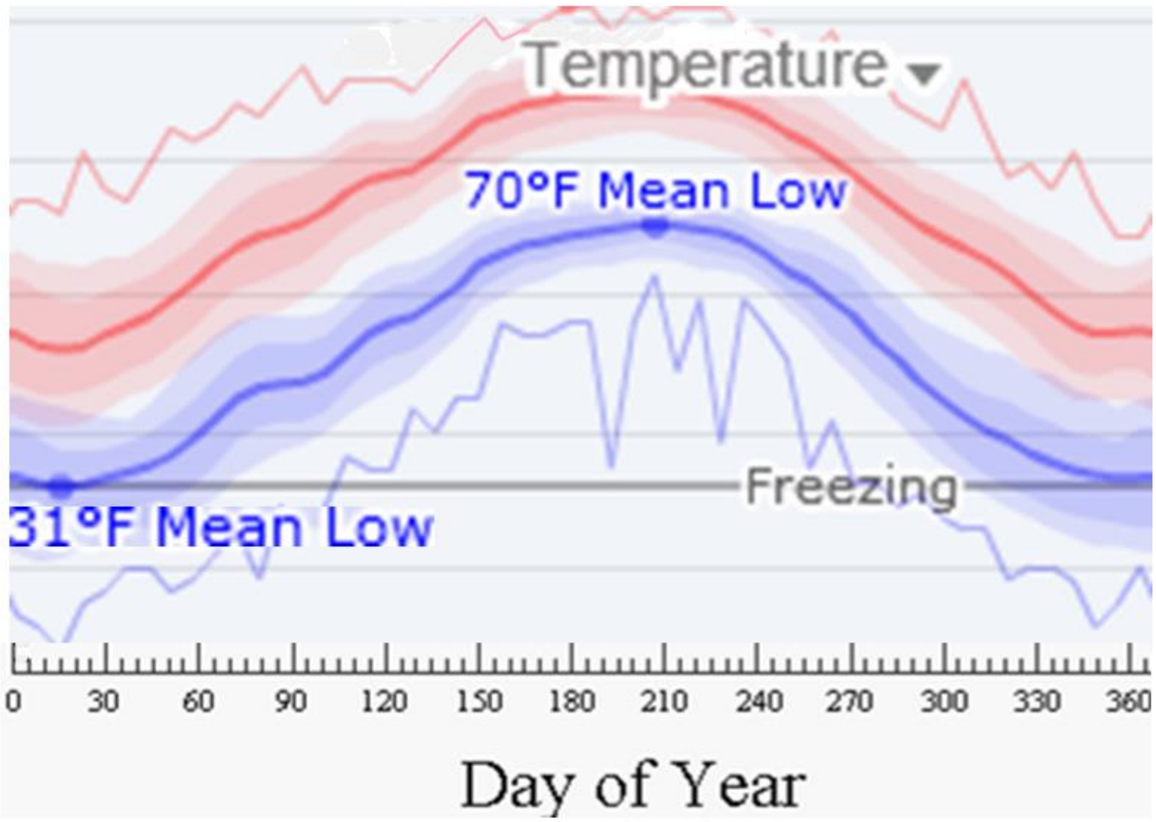


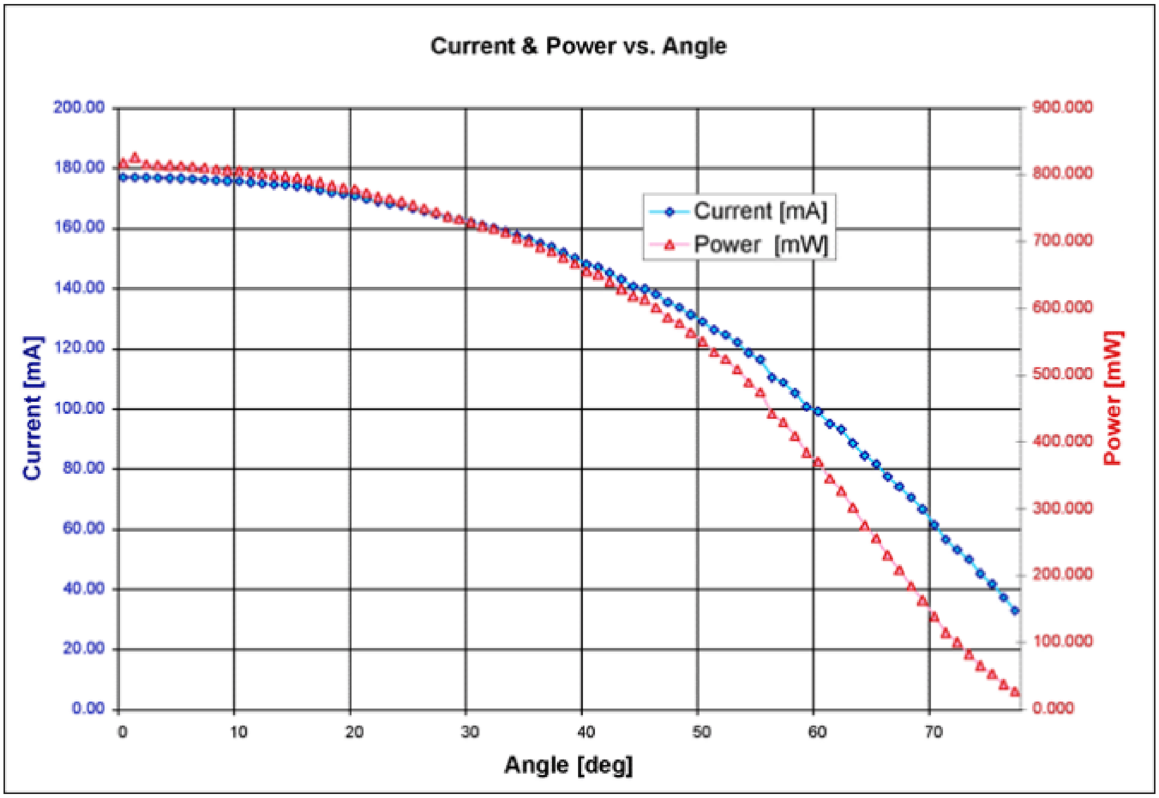
Image source: *Concord, NC Weather*. WeatherSpark, n.d. Web. 21 Feb. 2013 <http://weatherspark.com/#!dashboard;a=USA/NC/Concord>.

5. A solar cell will generate its maximum current when the cell is aligned to 0 degrees (perpendicular to the Sun's rays) and the cell will generate no current when the cell is aligned to 90 degrees (parallel to the Sun's rays). The **blue** scatterplot below gives the generated current in mA for various alignments in degrees. Do parts a – e as outlined above and then answer two more questions:

   f) According to your model, what is the maximum current this solar cell will generate?

   g) How much current will the cell generate at 45 degrees?

   h) What percentage of the maximum current will the cell generate at 45 degrees?



Source: *Solar Panel*. Rob Roberts, Camilo Jimenez, Naji Ghosseiri, n.d. Web. 23 Feb. 2013. <http://userwww.sfsu.edu/ozer/engr300-solar1N.pdf>.

6. In 3-5, can you conclude that you have found an <u>appropriate</u> model?

Problem Set 4-7

Complete steps a-d for each of the data sets below using Fathom.

a) If possible, find a sinusoidal modeling equation for the data and graph your function over the data plot in Fathom.

b) Give the period and amplitude of your model and explain what they mean within the context of the data.

c) Give the coordinates of the point (h, k) for your model and explain what they mean within the context of the data.

d) Make a residual plot and comment on the appropriateness of your model.

The following data sets are posted in
Course Material → 4-7: Modeling with Circular Functions

1. 💻 Average Monthly Temperature for Worcester, MA
Independent variable – Month number
Dependent variable – Average monthly temperature

2. 💻 Deerfield Sunrise and Sunset Times
Independent variable - Month number
Dependent variable - Daylight_minutes

3. 💻 Pizza Prices at UC Davis
Independent variable – Pizza diameter
Dependent variable - Price

4. 💻 Refrigerator Temperatures
Independent variable - Time
Dependent variable – Temperature

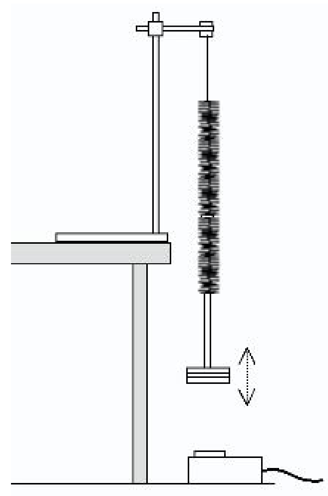
5. 💻 Satellite Orbit
Independent variable – Orbit time
Dependent variable - Distance from the Earth

In addition to parts a-d described above, answer one additional question.

e) According to your model (use the equation), how far north or south of the equator is the satellite 45 minutes into its orbit?

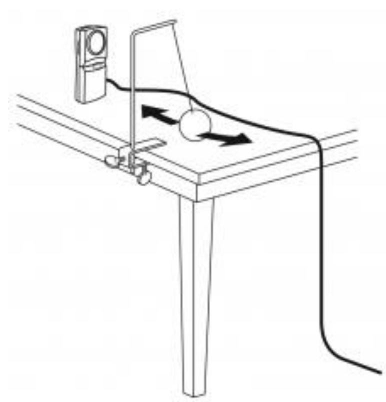
Problem Set 4-8

Data for 1 and 2 will be collected in class. Complete the same parts a-e that you did in problem set 4-6 for each situation, using Fathom to “analyze the data”. Be sure to include a residual plot.



1. 💻 Weight on a spring
   Independent variable – Time
   Dependent variable – Height (vertical distance from the sensor)

   Image source: *Investigating a Mass on a Spring*. Nuffield Foundation, n.d. Web. 22 Feb. 2013. <http://www.nuffieldfoundation.org/sites/default/files/images/ Investigating%20a%20mass-on-spring%20oscillator_322.jpg>.



2. 💻 Pendulum
   Independent variable – Time
   Dependent variable – Horizontal distance from the sensor

   Image source: *Pendulum Motion*. Vernier, n.d. Web. 22 Feb. 2013. <http://www.vernier.com/images/cache/ lab.RWV-22-DQ-tic_toc.424.238.png>.

3. 💻 CO_2 data 1980-1990.
   Independent variable – Year
   Dependent variable – CO_2 concentration
   Fit a model to the data. This will require you to “compose” two functions together. It is challenging, but you can do it.

Problem Set 4-9

All of the problems in this set come from the following source:
The North Carolina School of Science and Mathematics. *Contemporary Precalculus Through Applications*. 2nd Edition, 1999 ed. N.p.: Everyday Learning Corp., n.d. Print.

1. Suppose a Ferris wheel has radius 33.2 feet and makes three complete revolutions every minute. For clearance, the bottom of the Ferris wheel is 4 feet above the ground.
   a) Sketch a graph that shows how a particular passenger's height above the ground varies over time as he or she rides the Ferris wheel. Assume that the passenger is at the bottom of the Ferris wheel when $t = 0$.
   b) Write a function that models the passenger's height above the ground as a function of time.

2. In Los Angeles on the first day of summer (June 21) there are 14 hours and 26 minutes of daylight, and on the first day of winter (December 21) there are 9 hours and 54 minutes of daylight. On average there are 12 hours and 10 minutes of daylight; this average amount occurs on March 20 and September 22.
   a) Sketch a graph that displays this information. Use number of months after June 21 as the independent variable and minutes of daylight as the dependent variable.
   b) Write an equation that expresses d, the amount of daylight per day in minutes, as a function of t, the number of months after June 2.

3. A population of lynx oscillates in a four-year cycle. Kate is a biologist who keeps records of the lynx population in a small area. She has counted lynx on January 1st of each year. Her first count in 1994 was 40; in 1995, 60; in 1996, back to 40; and in 1997, 20. The population was back to 40 in 1998.
   a) Sketch a graph that shows the lynx population as a function of time. Assume that the relationship is sinusoidal.
   b) Write a function that expresses the lynx population as a function of time.

4. In a tidal river, the time between high tide and low tide is approximately 6.2 hours. The average depth of the water at a port on the river is 4 meters; at high tide the depth is 5 meters.
   a) Sketch a graph of the depth of the water at the port over time if the relationship between time and depth is sinusoidal and there is a high tide at noon.
   b) Write an equation for your curve. Let t represent the number of hours after noon.