Chapter 6 – Sequences and Series

Problem Set 6-0

1. For each of the following sequences describe the sequence in words and then write standard recursive notation:

a. 17, 20, 23, 26, 29, 32, ….

The first term is 17 and each term after the first is 3 more than the previous term.

$$\begin{cases} a_1 = 17 \\ a_n = a_{n-1} + 3, n > 1 \end{cases}$$

b. 100, 90, 80, 70, 60, 50, …

The first term is 100 and each term after the first is 10 less than the previous term.

$$\begin{cases} a_1 = 100 \\ a_n = a_{n-1} \text{ -}10, n > 1 \end{cases}$$

c. 4x, 5x, 6x, 7x, 8x, …

The first term is 4x and each term after the first is x more than the previous term.

$$\begin{cases} a_1 = 4x \\ a_n = a_{n-1} + x, n > 1 \end{cases}$$

d. 8, 16, 32, 64, 128, 256, …

The first term is 8 and each term after the first is 2 times the previous term.

$$\begin{cases} a_1 = 8 \\ a_n = a_{n-1} \cdot 2, n > 1 \end{cases}$$

e. 81, 27, 9, 3, 1, 1/3, …

The first term is 81 and each term after the first is 1/3 of the previous term.

$$\begin{cases} a_1 = 81 \\ a_n = a_{n-1} \cdot \dfrac{1}{3}, n > 1 \end{cases}$$

f. 10, 100, 10,000, 100,000,000, 10,000,000,000,000,000, …

(10^1, 10^2, 10^4, 10^8, 10^{16}, …)

The first term is 10 and each term after the first is the sqaure of the previous term.

$$\begin{cases} a_1 = 10 \\ a_n = (a_{n-1})^2, n > 1 \end{cases}$$

2. Consider the sequence defined by $\begin{cases} a_1 = 7 \\ a_n = a_{n-1} + 4, n > 1 \end{cases}$.

   a. Write the first 4 terms of the sequence.
      7, 11, 15, 19

   b. What is the 113th term of the sequence?

position	term
1	7
2	11
113	455

=↑ + 4

The 113th term is 455

3. Consider the sequence defined by $\begin{cases} b_1 = 8{,}000 \\ b_n = b_{n-1} \cdot 1.007 - 329, n > 1 \end{cases}$.

   a. Write the first 4 terms of the sequence (rounded to the nearest hundredth).
      8000, 7727, 7452.09, 7175.25

   b. What is the 21st term of the sequence?

position	term
1	8,000
2	7,727
21	2,161.20

=↑ * 1.007 - 329

The 21st term is about 2,161.2

4. Consider the sequence defined by $\begin{cases} b_1 = 180^0 \\ b_n = \sin(b_{n-1}), n > 1 \end{cases}$.

   a. Write the first 4 terms of the sequence.
      180, 0, 0, 0

   b. What is the 1,000th term of the sequence?
      0. Since the sin(0) = 0, every term starting with the second is 0.

5. Jo Ann is learning Chinese. Currently she has memorized the meaning of 400 characters. Each month she forgets 10% of the characters she has learned (she remembers 90%) and learns 300 new characters.
   a. Write standard recursive notation for the sequence that represents the number of characters she knows each month.

$$\begin{cases} b_1 = 400 \\ b_n = 0.9 \cdot b_{n-1} + 300, n > 1 \end{cases}$$

   b. If she continues to forget and learn as described above, what is the maximum number of characters she can learn? Explain why.

position	term
1	400
2	660
82	2,999
83	3,000

=↑ * 0.9 + 300

The sequence approaches 3,000 so she can learn a maximum of 3,000 characters

10% of 3000 is 300, so when she knows 3000 characters during the next month she forgets 300 characters and then learns 300.

Problem Set 6-1

1. Go back to the sequences in 6-0-1. For each sequence, state whether it is arithmetic, geometric, or neither. If it is arithmetic or geometric, write the explicit form.
   a. **Arithmetic;** $a_n = 17 + 3 \cdot (n-1)$
   b. **Arithmetic;** $a_n = 100 - 10 \cdot (n-1)$
   c. **Arithmetic;** $a_n = 4x + x \cdot (n-1)$
   d. **Geometric;** $g_n = 8 \cdot 2^{n-1}$
   e. **Geometric;** $g_n = 81 \cdot \left(\frac{1}{3}\right)^{n-1}$
   f. **neither**

2. Consider the arithmetic sequence defined by $\begin{cases} a_1 = 12 \\ a_n = a_{n-1} + 3, n > 1 \end{cases}$.
   a. Is the sequence defined explicitly or recursively? **recursive**
   b. What does a_{n-1} mean? **the previous term**
   c. What is the first term and constant difference? **12, 3**
   d. Write the first 4 terms of the sequence. **12, 15, 18, 21**
   e. What is the 100th term of the sequence? **12+3*99 = 309**

3. Consider the geometric sequence defined by $g_n = 3 \cdot 10^{n-1}$.
   a. Is the sequence defined explicitly or recursively? **explicit**
   b. What does g_n mean? **this term or the current term**
   c. What is the first term and constant ratio? **3, 10**
   d. Write the first 4 terms of the sequence. **3, 30, 300, 3000**
   e. What is the 10th term of the sequence? **$3(10)^9$ = 3,000,000,000**

4. A private school that has more applicants than positions decides to let the size of the student body grow each year by 2%. Give an adjective that describes how the student body grows over time. **"Geometrically" describes the growth of the student body because we multiply by a constant each year to next year's number of students in the student body.**

5. For the arithmetic sequence a_1=1.79, a_2=1.82, a_3=1.85, a_4=1.88, …
   a. Write a recursive formula a_n.
   $\begin{cases} a_1 = 1.79 \\ a_n = a_{n-1} + 0.03, n > 1 \end{cases}$
   b. Write an explicit formula a_n.
   $a_n = 1.79 + 0.03 \cdot (n-1)$

6. For the geometric sequence g_1=20, g_2=4, g_3=0.8, g_4=0.16, …

a. Write a recursive formula g n.

$$\begin{cases} g_1 = 20 \\ g_n = g_{n-1} \cdot 0.2, n > 1 \end{cases}$$

b. Write an explicit formula g_n.

$$g_n = 20 \cdot 0.2^{n-1}$$

7. The information about the following sequence refers to either an arithmetic or geometric sequence. Write both a recursive and explicit formula for each sequence.

a. b_3=12, b_4=1.2, b_5=0.12

$$\begin{cases} b_1 = 1{,}200 \\ b_n = b_{n-1} \cdot 0.1, n > 1 \end{cases}$$

$$b_n = 1{,}200 \cdot 0.1^{n-1}$$

b. c_2=106, c_3=89, c_4=72

$$\begin{cases} c_1 = 123 \\ c_n = c_{n-1} - 17, n > 1 \end{cases}$$

$$c_n = 123 - 17 \cdot (n-1)$$

c. d_2=33.6, d_4=86.016, d_6= 220.20096

$$\begin{cases} d_1 = 21 \\ d_n = d_{n-1} \cdot 1.6, n > 1 \end{cases}$$

$$d_n = 21 \cdot 1.6^{n-1}$$

8. The sequence below is not a geometric sequence, but the numbers increase very close to geometrically. Write both a recursive and explicit formula that defines a geometric sequence that has the first four terms very close to the ones given. The first four terms of the sequence are 31, 47, 70, 105.

$$\begin{cases} g_1 = 31 \\ g_n = g_{n-1} \cdot 1.5, n > 1 \end{cases}$$

$$g_n = 31 \cdot 1.5^{n-1}$$

You can get 1.5 by taking the three ratios between consecutive terms (47/31, 70/47, and 105/70) and averaging them. The average is close to 1.5

Problem Set 6-2

1. This summer you have exactly 6 weeks free to work and both your Mom and your Dad have offered you a job. Your Mom offers you 1 penny today, 2 pennies tomorrow, 4 pennies the next day, growing geometrically for 6 weeks. Your Dad offers you $1,000 the first day, $1,100 the second day, $1,200, growing arithmetically for 6 weeks. Assume you work 6 weeks X 5 days per week (30 days total). Use the arithmetic series formula to calculate the total amount you would earn from your **father**.

$$\begin{cases} f_1 = 1{,}000 \\ f_n = f_{n-1} + 100, n > 1 \end{cases}$$

or

$$f_n = 1{,}000 + 100(n-1)$$

So, $f_{30} = 1{,}000 + 100(30-1) = 3{,}900$

and, $S_{30} = \left(\frac{30}{2}\right)(1{,}000 + 3{,}900) = \$73{,}500$

2. Given an arithmetic sequence a_n, an arithmetic series can be defined by $\begin{cases} S_1 = a_1 \\ S_n = a_n + S_{n-1}, n > 1 \end{cases}$, where S_n is the sum of the first n terms. Use this fact to generate a table of n vs. S_n with Excel. The values in the S_n column are called the partial sums. Use this table to verify your answer in (1).

Day	Pay that day	Total pay
1	$1,000	$1,000
2	$1,100	$2,100
30	$3,900	$73,500

= ↑ + 100

= ↑ + ←

Total after 30 days is $73,500

3. Given an arithmetic sequence a_n, another way to define the arithmetic series is $S_n = \sum_{i=1}^{n} a_i$.

a. If a_n=2 + 3(n-1), find $\sum_{i=1}^{90} a_i$.

(90/2) · (2 + 269) = 12,195

b. If $\begin{cases} b_1 = -17 \\ b_n = b_{n-1} + 9, n > 1 \end{cases}$, find $\sum_{i=1}^{1000} b_i$.

4,478,500

c. If c_n is an arithmetic sequence and c_2=106, c_3=89, c_4=72, find $\sum_{i=1}^{25} c_i$.

-2,025

4. A family has 5 children ages 4, 6, 8, 10, and 12. The parents have $100 that they are willing to give the children for allowance each month and are thinking of how they should divide up the $100. How much does each child get if the amount that they get is proportional to their age. (For example, the kid who is 12 will get twice as much as the kid who is 6.)
   $4x + 6x + 8x + 10x + 12x = 100$
   $40x = 100$, so $x = 2.5$.
   The children will get $10, $15, $20, $25, $30

Problem Set 6-3

1. This summer you have exactly 6 weeks free to work and both your Mom and your Dad have offered you a job. Your Mom offers you 1 penny today, 2 pennies tomorrow, 4 pennies the next day, growing geometrically for 6 weeks. Your Dad offers you $1,000 the first day, $1,100 the second day, $1,200, growing arithmetically for 6 weeks. Assume you work 6 weeks X 5 days per week (30 days total). Use the geometric series formula to calculate the total amount you would earn from your **mother**.

$$\frac{0.01\cdot\left(1-2^{30}\right)}{1-2}=\$10{,}737{,}418.23$$

2. Given a geometric sequence g_n, a geometric series can be defined by

$$\begin{cases} S_1 = g_1 \\ S_n = g_n + S_{n-1}, n > 1 \end{cases}$$

where S_n is the sum of the first n terms.

a. Use this fact to generate a table of n vs. S_n with Excel. Verify your answer in (1) with this table.

=↑ * 2

=↑ + ←

Day	Pay that day	Total pay
1	$ 0.01	$ 0.01
2	$ 0.02	$ 0.03
30	$ 5,368,709.12	$10,737,418.23

Total after 30 days is $10,737,418.23

b. Use Excel to find out on which day the total you receive from your mother exceeds what you received from your father.

Day	Dad-today	Dad-so far	Mom-today	Mom-so far	Dad-Mom so far
1	$1,000.00	$1,000.00	$0.01	$0.01	$999.99
2	$1,100.00	$2,100.00	$0.02	$0.03	$2,099.97
22	$3,100.00	$45,100.00	$20,971.52	$41,943.03	$3,156.97
23	$3,200.00	$48,300.00	$41,943.04	$83,886.07	-$35,586.07
30	$3,900.00	$73,500.00	$5,368,709.12	$10,737,418.23	-$10,663,918.23

On the 23rd day, Mom's total exceeds Dad's total by $35,586.07.
On the 30th day, Mom's total exceeds Dad's total by $10,663,918.23.

3. Given a geometric <u>sequence</u> g_n, another way to define the geometric <u>series</u> is $S_n = \sum_{i=1}^{n} g_i$.

   a. If $a_n = 2 \cdot 3^{n-1}$, find $\sum_{i=1}^{15} a_i$.

      14,348,906

   b. If $\begin{cases} b_1 = 2.5 \\ b_n = b_{n-1} \cdot 4, n > 1 \end{cases}$, find $\sum_{i=1}^{8} b_i$.

      54,612.5

   c. If c_n is a geometric sequence and $c_1=.9$, $c_2=.09$, $c_3=.009$, find $\sum_{i=1}^{10} c_i$.

      0.9999999999

4. A family is going on a 600 mile trip and Dad says he is willing to make 5 stops, so they will break the 600 miles into 6 "legs". Dad's thinking is that as the trip drags on, the family will need to take breaks more and more frequently, so he decides to make the number of miles traveled in each leg a non-increasing geometric sequence.

   a. If Dad decides he wants to travel 80% of the previous leg on each leg, how far will they travel on each leg? Use an explicit equation to answer this question and perhaps verify with Excel. (Hint: your answer must be a decreasing geometric series whose sum is 600.)

      $$S_n = \frac{g_1(1-r^n)}{(1-r)} \quad \to \quad 600 = \frac{g_1(1-r^6)}{(1-r)} \quad \to \quad g_1 = \frac{600 \cdot (1-r)}{(1-r^6)}$$

      a. $g_1 = \frac{600 \cdot (1-0.8)}{(1-0.8^6)} \approx 162.6$

      162.6, 130.1, 104.1, 83.3, 66.6, 53.3

   b. If the family stops after 180 miles (the first leg is 180 miles), how far will they travel on each leg? Hint: Use Excel to solve this question by setting up a spreadsheet where "r" is a named cell. Vary "r" until the sum is 600 and find "r" to the nearest three decimal places.

	A	B	C	D	E
1	Leg	Distance		ratio	
2	1	180.000		0.756	
3	2	136.080			
4	3	102.876			
5	4	77.775			
6	5	58.798			
7	6	44.451			
8	sum	599.980			

=B2*ratio

this cell named "ratio"

Change cell D2 (trial and error) until B8 ≈ 600 or use the Excel feature called "Solver".

5. Find the sum of the geometric series: 3600 + 900 + 225 + … + 0.87890625
   4,799.70703125

6. Given the sequence 7, 7, 7, 7, 7, … write both a recursive arithmetic definition for the sequence and a recursive geometric definition for the sequence.

$$\begin{cases} a_1 = 7 \\ a_n = a_{n-1} + 0, n > 1 \end{cases}$$

$$\begin{cases} g_1 = 7 \\ g_n = g_{n-1} \cdot 1, n > 1 \end{cases}$$

<u>Problem Set 6-4</u>

1. When buying a car, unless you pay the entire amount at the time of delivery, you make a down-payment and borrow the rest from a bank. Typically the money is paid back in equal monthly installments over a number of years. The amount of the loan that you owe the bank each month is equal to much you owed the bank the previous month times the monthly interest rate minus the payment.

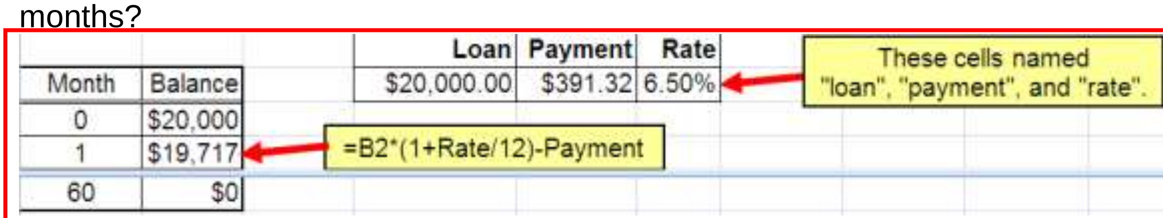
   The sequence can be described recursively by:

$$\begin{cases} L_0 = \text{Loan} \\ L_n = L_{n-1} * (1 + \text{rate}/12) - \text{Payment}, n > 0 \end{cases}$$

   where L_n is the amount you owe the bank at the end of month n. Assume you make your payment at the end of the month.

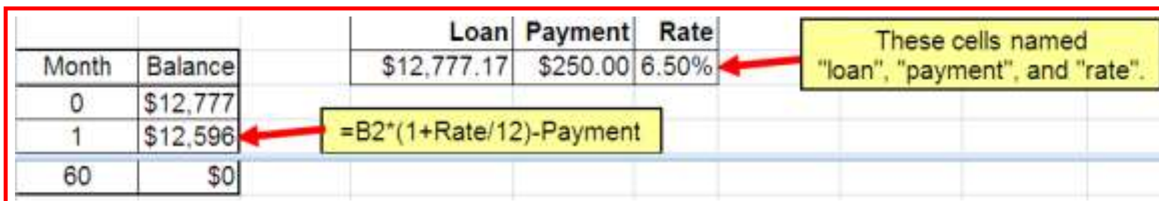
   The most common number of months to pay back the loan (called the term of the loan) is 60 months (5 years).

   a. If you want to borrow $20,000 and the annual interest rate is 6.5%, what is the monthly payment (to the nearest penny) if you pay the loan back in 60 months?



   The monthly payment would be $391.32. Go to the web and search for "car loan calculator". Verify your answer to this question and the next one.

   b. If you can afford $250 per month and the annual interest rate is 6.5%, how much of a loan will you get if you pay the loan back in 60 months?
   Note: this question is probably more realistic than part (a). When you buy your first car, it is probably at a time when you do not have that much money. You would see how much money you can afford for a car each month after paying all your bills and <u>then</u> you would see how big a car loan you can afford, given the monthly payment you can afford.



   The loan would be $12,777.17.

Problem Set 6-5

1. How much money does a typical US household need to retire? We will take two passes at this question. First, we will assume the savings are not in the market, then we will assume that the savings are in the market.

This is approximately a one week project. Each night you are to do the following:

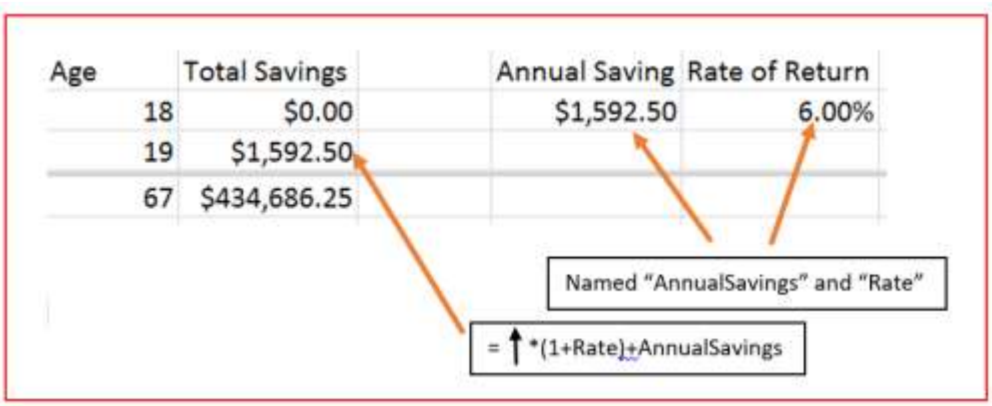
a. Create a Word document that documents 45 minutes of research. The Word document should have citations for your sources and the citations should be in MLA form.
   Use http://www.noodletools.com/noodlebib/citeone_s.php?style=MLA

b. For each source, write one thing you learned and one question of something you don't understand from that source. See example below.

c. Answer the question – How much money does a typical US household need to retire? Give a dollar value. The first night this might be challenging, but do your best.

An example of a citation, something learned, and a question:
Retirement. CNN Money, n.d. Web. 1 Feb. 2015.
<http://money.cnn.com/retirement/>.
The average 401(k) balance in the US is currently $91,300.
What is a 401(k)?

2. Many people buy a cup of coffee and pastry on the way to work every day. Let's say they work from age 18 to age 67, work 49 weeks per year, and 5 days per week. Each day they buy a coffee and pastry for $6.50.
   a) How much money did they spend per year on coffee?
      49*5*$6.50 = $1,592.50
   b) How much would they have at age 67 if they did not buy the coffee and instead each year invested the money in the market earning 6.0% rate of return?



Problem Set 6-6

Tim and Tom Save Money For Retirement

Adapted from…

Source: Contemporary Precalculus through Applications, NCSSM, page 186, problem 7

Original Source: Smith, Keith, "Tim and Tom's Financial Adventure" Pull-Out, consortium, Number 38, Fall 1991, COMAP, Inc., Lexington, MA

Additional Source: Phillip Rash, Handout from TSM Talk, Feb 2009

Tim and Tom were twins. They both went to work right out of college with identical jobs, identical salaries, and at the end of each year, they received identical bonuses of $2,000 which happened to be paid out on their birthday. They each received their first bonus on their 22nd birthday.

Tim was conservative and concerned about his future. Each year, starting on his 22nd birthday, he invested his $2,000 bonus in an IRA (Individual Retirement Account) earning 8.5% rate of return compounded annually. He kept investing $2,000 per year up to and including his 32nd birthday, when he made his last payment into the investment plan. On his 33rd birthday Tim figured he had saved enough and wanted to have some fun. He spent the $2,000 bonus he got on his 33rd birthday on a vacation to the Bahamas. He enjoyed it so much that he continued spending his yearly $2,000 bonus on a vacation to the Bahamas every year up to and including his 65th birthday on which he took his vacation and then started his retirement.

Tom, on the other hand, liked to party and believed that life was too short to be concerned about saving for the future. On his 22nd birthday he spent his $2,000 bonus on a vacation in the Bahamas. In fact he spent his $2,000 bonus every year on a Bahamas vacation. Until… As Tom's 33rd birthday approached, his brother Tim told him how much his IRA (with interest) had grown to. On his 33rd birthday, Tom invested his $2,000 bonus in an IRA earning 8.5% rate of return compounded annually. He got nervous that he hadn't saved enough, so he continued investing his yearly bonus in the same plan until his 65th birthday on which he put the last $2,000 bonus in his IRA and then retired.

Create an Excel spreadsheet that will help you answer the following questions. You must (as usual) show all your work and clearly communicate how you got your answers.

1. After their 65th birthday,
   a. How much money had Tim contributed to his IRA?
   b. How much money had Tom contributed to his IRA?
   c. How much more money had Tom contributed than Tim?
   d. How much money had Tim accumulated in his IRA?
   e. How much money had Tom accumulated in his IRA?
   f. Who had more accumulated in his IRA? How much more?

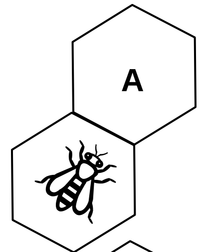
2. Assume that neither retired at age 65. If Tim kept going to the Bahamas with his bonus and Tom kept investing his bonus into his IRA, how long would it take until Tom's savings equaled Tim's?

3. How much would Tom need to have invested into the IRA each year (from his 33rd birthday through and including his 65th birthday) so that the twins would have the same amount in their IRAs at age 65?

4. If everything else stays the same, what rate of return results in Tim and Tom having the same final balance when they retire?

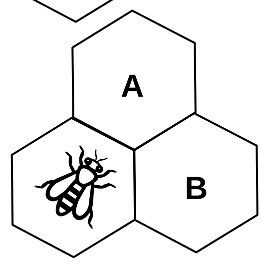
Problem Set 6-7

1. * Consider a bee in a honeycomb that is moving to the right. The bee can take any path to the end of the honeycomb as long as it doesn't move to the left.
   a. As the bee moves from its initial position to A, how many paths can it take and what is the path?



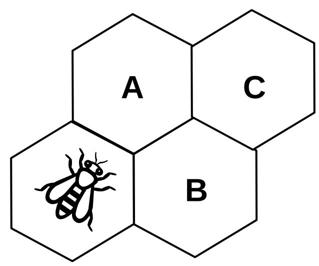
      Answer: 1; A

   b. As the bee moves from its initial position to B, how many paths can it take and what are the paths?

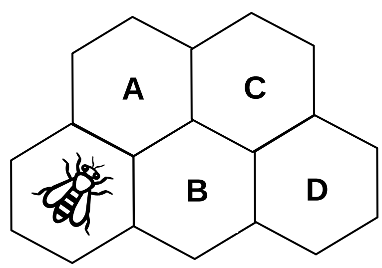


      Answer: 2; AB,B

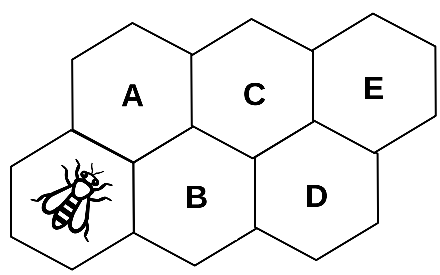
   c. As the bee moves from its initial position to C, how many paths can it take and what are the paths? (Note: BAC is not a valid path because the move from B to A is backwards.)



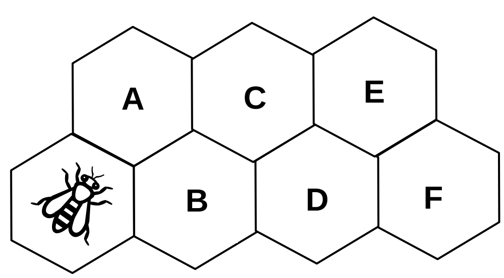
   d. As the bee moves from its initial position to D, how many paths can it take and what are the paths?



   e. As the bee moves from its initial position to E, how many paths can it take and what are the paths?



   f. As the bee moves from its initial position to F, how many paths can it take and what are the paths?



2. * A machine sells tokens for 25¢ and accepts quarters and half dollars. There is only 1 way to buy 1 token – put in one quarter. Call that "sequence" Q. There are 2 ways to buy 2 tokens – put in 2 quarters (QQ) or one half dollar (H). For the purposes of this problem order is important so QQH is different form QHQ.
   a. How many sequences exist to buy 3 token? What are the sequences?
   b. How many sequences exist to buy 4 token? What are the sequences?
   c. How many sequences exist to buy 5 token? What are the sequences?
   d. How many sequences exist to buy 6 token? What are the sequences?
   e. Without listing the sequences, how many sequences do you think exist to buy 7 token?

3. One of the most famous and fascinating sequences is the Fibonacci Sequence.

   It is defined recursively as $\begin{cases} f_1 = 1 \\ f_2 = 1 \\ f_n = f_{n-2} + f_{n-1}, n > 2 \end{cases}$

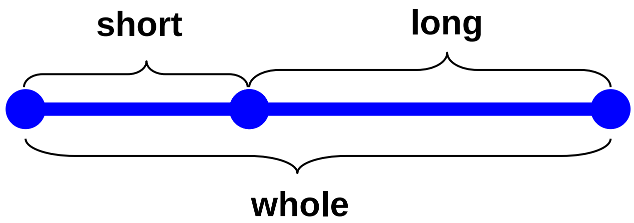
   Generate the Fibonacci Sequence in Excel in column A. In column B, generate the ratio between consecutive terms, $\frac{f_n}{f_{n-1}}$.

   Use this information to estimate $\lim_{n \to \infty} \frac{f_n}{f_{n-1}}$.

* The idea for problem 1 and 2 above come from:
Fascinating Fibonaccis, Chapter 6, Trudi Garland, Dale Seymour, 1987

Problem Set 6-8

1. When a segment is divided into two segments such that $\frac{\text{short}}{\text{long}} = \frac{\text{long}}{\text{whole}}$, the parts are said to be in Golden Proportion. Let short = 1, long = x, and solve for x. (Hint: you will need the quadratic formula).



$$\frac{\textbf{short}}{\textbf{long}} = \frac{\textbf{long}}{\textbf{whole}}$$

2. Consider the sequence $\begin{cases} p_1 = 1 \\ p_n = 1 + 1/p_{n-1}, n > 1 \end{cases}$.

   Estimate $\lim_{n \to \infty} p_n$ by generating the sequence with Excel and see what p_n appears to converge to as n increases.

Problem Set 6-9

1. *Annual plants like the sunflower grow from seed in the spring, mature over the summer, and die completely in the fall. They must produce seeds for the following year in order for the species to survive.

    a. In the first 4 weeks of growth, a sunflower plant grows quickly. Although the height does not increase exactly geometrically, it increases very close to geometrically. Give a recursively defined geometric sequence that models the growth of the sunflower plant given in the table below. The sequence will not fit the data exactly, but it is possible to give a recursively defined geometric sequence that models the data very closely. (Note: You can copy tables in Word documents and paste them into Excel.)

week	height (cm)
1	21
2	36
3	58
4	88

    b. Based on your sequence in (a), what is the growth factor and the growth rate?

    c. A sunflower matures in about 12 weeks. Look at the sunflower height data for the entire 12 weeks. Create a scatterplot of the data. From the 11th to the 12th week, what is the growth factor and the growth rate?

week	height (cm)
1	21
2	36
3	58
4	88
5	126
6	170
7	206
8	231
9	246
10	251
11	253
12	255

Data Source: The idea came from: Quantitative Learning Project, QELP Data Set 009, http://www.seattlecentral.org/qelp/sets/009/009.html, 3/21/03

    d. Use Excel to create a third column which gives the growth rate from week n to week n+1.

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e. If the growth rate were constant then the relationship between last week's height, h_{n-1}, and this week's height, h_n, would be $h_n = h_{n-1}\cdot(1+rate)$. In this case, however, the rate is not constant so we need to multiply the rate by a new logistic parameter which gradually changes from 1 (100%) to 0 (0%) during the 12 weeks.

So,
$h_n = h_{n-1}\cdot(1+rate)$
becomes
$h_n = h_{n-1}\cdot (1+rate(\text{logistic parameter}))$.

The logistic parameter that we use is $\left(1-\frac{h_{n-1}}{h_{max}}\right)$, where h_{max} is the maximum height.

At the beginning of growth,

h_{n-1} is small,	$\frac{h_{n-1}}{h_{max}}$ is close to 0,	$\left(1-\frac{h_{n-1}}{h_{max}}\right)$	is close to 1

Near the end of growth,

h_{n-1} is large,	$\frac{h_{n-1}}{h_{max}}$ is close to 1,	$\left(1-\frac{h_{n-1}}{h_{max}}\right)$	is close to 0

f.

The recursive relationship becomes $h_n = h_{n-1}\bullet\left[1+rate\bullet\left(1-\frac{h_{n-1}}{h_{max}}\right)\right]$. This is called a recursively defined logistic model which has a distinctive "S" shape graph. Create a new column in the spreadsheet that represents the predicted height using the recursive relationship above. Adjust rate and h_{max} until the logistic model closely matches the scatterplot.

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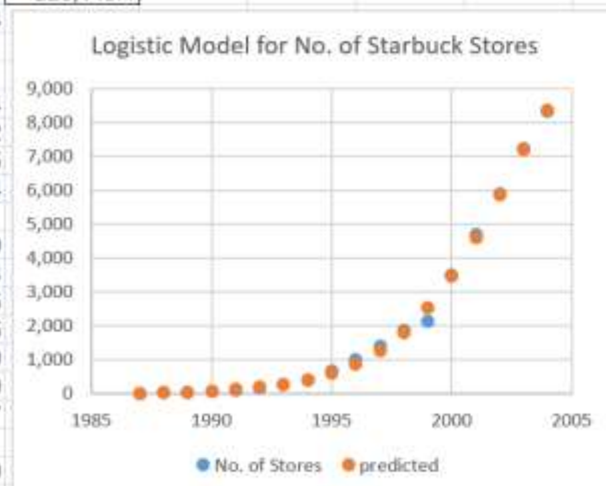
2. Find a recursively defined, logistic model for the number of Starbuck Stores between 1987 and 2004. Copy and paste the data below into Excel.

   a. Write a recursively defined logistic model for the data.
   b. Use the idea of “least squares” and the “Solver” tool on Excel to minimize the sum of the squares of the errors by changing the initial number of stores (in your model), the maximum number of stores (in your model – also called the carrying capacity), and the growth rate (of your model). According to your model, what is the maximum number of stores, worldwide that Starbucks can expect to have?

	A	B	C	D	E
1	Year	No. of Stores	predicted	error^2	sum
2	1987	17	28.15703	124.4793	223,149.1
3	1988	33	41.53981	72.92834	
4	1989	55	61.25925	39.17821	
5	1990	84	90.28748	39.53241	
6	1991	116	132.9576	287.5604	
7	1992	165	195.5483	933.1992	
8	1993	272	287.0745	227.2405	
9	1994	425	420.3027	22.06464	
10	1995	676	612.9381	3976.801	
11	1996	1,015	888.755	15937.79	
12	1997	1,412	1278.082	17934.04	
13	1998	1,886	1816.43	4840.035	
14	1999	2,135	2539.232	163403.5	
15	2000	3,501	3470.252	945.4469	
16	2001	4,709	4602.857	11266.39	
17	2002	5,886	5879.576	41.26337	
18	2003	7,225	7185.67	1546.871	
19	2004	8,337	8375.869	1510.829	
20					
21	Initial	28.2			
22	Max	11,014.2			
23	Rate	47.7%			



Starbucks Store Data, Starbuck, http://www.starbucks.com/aboutus/timeline.asp, 2/12/2006

11,014 Stores

3. You might have been told in the past that populations grow geometrically (exponentially). Why do they ***not*** actually grow geometrically (exponentially) in the long run?
   Geometric (exponential) growth is not sustainable in an environment with limited resources.