7 – Exponential and Logarithmic Functions

Albert Einstein – "The most powerful force in the universe is compound interest".

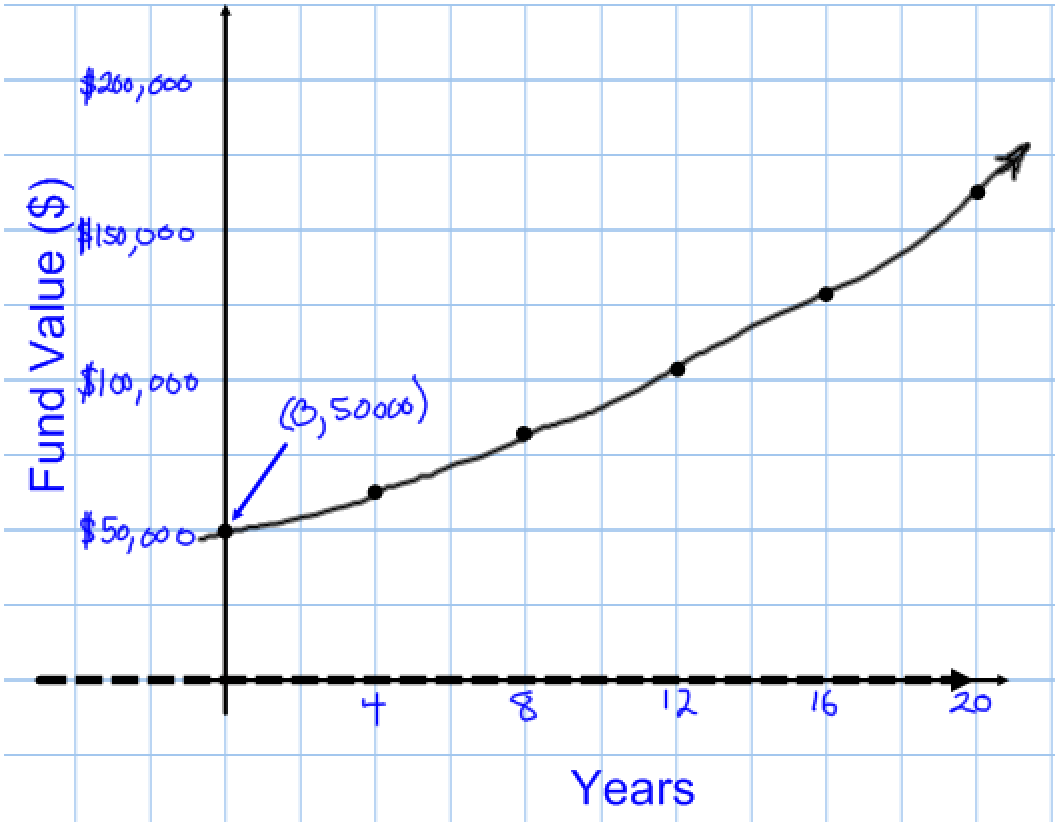
Problem set 7-1

1. On you calculator type 100000000000 and then ENTER. What is on the display of your calculator? What does it mean?
   The calculator displays 1E11 which represents 1×10^{11}. 1E11 is calculator form and not standard mathematical notation. Calculator scientific notation is NOT acceptable when communicating mathematical work.

2. On you calculator type 1/1000000 and then ENTER. What is on the display of your calculator? What does it mean?
   The calculator displays 1E-6 which represents 1×10^{-6}.

3. Instead of making a down payment on a house, a couple that lives in an apartment decides to invest \$50,000 that they inherited from Aunt Zelda into a real estate fund that earns 6.3% interest, compounded annually. Let *A* be the value of the fund after *t* years.
   a. Write *A* as a function of *t*.
      $A = \$50{,}000 \cdot 1.063^{t}$
   b. What will the value of the investment be after 10 years? 20 years?
      $A(10) = \$50{,}000 \cdot 1.063^{10} \approx \$92{,}109.12$
      $A(20) = \$50{,}000 \cdot 1.063^{20} \approx \$169{,}681.81$

c. Graph the function you wrote in (a) for years 0 through 20. Label the coordinates of the y-intercept and indicate all asymptotes with a dashed line.



d. Redo part (b), but use quarterly compounding

$$A(10) = \$50{,}000 \cdot \left(1+\left(\frac{.063}{4}\right)\right)^{4*10} \approx \$93{,}420.73$$

$$A(20) = \$50{,}000 \cdot \left(1+\left(\frac{.063}{4}\right)\right)^{4*20} \approx \$174{,}548.65$$

4. A couple has a baby and they want to put money in a college savings plan that assures them 7% interest for the next 18 years. If the parents want to have $125,000 when their child starts college, how much do they need to put in this college savings program now? Note: For this problem and the rest of the course, if no mention is made of the type of growth (annually, compounded quarterly, continuously, etc.) assume annual growth.

$$\$125{,}000 = P \cdot 1.07^{18}$$

$$P = \frac{\$125{,}000}{1.07^{18}}$$

The couple will need $36,982.99

5. Redo (4), but assume the interest is compounded continuously.

   $\$125{,}000 = P \cdot e^{0.07 \cdot 18}$

   $P = \dfrac{\$125{,}000}{e^{0.07 \cdot 18}}$

   The couple will need $35,456.75

6. In order for consumers to be able to compare interest rates from one institution to the next, the government often requires that the Effective Annual Yield (EAY) be reported. The EAY is the annualized interest rate which comes from the $\left(1 + \frac{r}{n}\right)^{n}$ portion of $A = P \cdot \left(1 + \frac{r}{n}\right)^{n\,t}$.

   EAY for interest compounded n times per year is $\left(1 + \frac{r}{n}\right)^{n} - 1$.

   EAY for interest compounded continuously is $e^{r} - 1$.

   Which interest rate has the highest EAY? The lowest?

   a. 6.5% compounded annually **6.5%** **lowest**
   b. 6.4% compounded monthly **≈ 6.5911%** **highest**
   c. 6.3% compounded continuously **≈ 6.5027%**

7. As a benchmark, people are often interested in how long it will take to double their money in a particular investment.
   a. Do the necessary calculations to fill in the table below:

Interest Rate Compounded Annually	Years to Double Your Money (round to the nearest year)	Product of Rate and Years to Double
3.5%	**≈ 20**	**≈ 70**
7.0%	**≈ 10**	**≈ 70**
10.0%	**≈ 7**	**≈ 70**

   b. State a rule of thumb for doubling your money.
   When the interest rate times the number of years equals about 70, the investment will have doubled. This is called the Rule of 70 (also called the Rule of 69 and Rule of 72). In business and finance, people often use the Rule of 70 to estimate how long it will take money to double.

8. 💻 Use Excel to find 1/0! + 1/1! + 1/2! + 1/3! + …
   Note: 5! = 5·4·3·2·1. 5! is read 5 factorial.
   In Excel, the formula for 5! Is =fact(5).

	A	B	C
1	position	term	partial sum
2	0	1	1
3	1	1	2
4	2	0.5	2.5
5	3	0.166667	2.666666667
6	4	0.041667	2.708333333
7	5	0.008333	2.716666667
8	6	0.001389	2.718055556
9	7	0.000198	2.718253968
10	8	2.48E-05	2.71827877
11	9	2.76E-06	2.718281526
12	10	2.76E-07	2.718281801
13	11	2.51E-08	2.718281826
14	12	2.09E-09	2.718281828
15	13	1.61E-10	2.718281828

Does this number look familiar? It should.

Problem set 7-2

The PowerPoint presentation called "Exponential Models: Discrete vs. Continuous" might be helpful for the following problems.

1. Use the on-line CIA World Factbook (Search "CIA World Factbook" with Google) to find the current population and annual population growth rate of Western Sahara.
   a. What will the population be in one year from the most current estimate?
      $513{,}000 \cdot 1.03^1 \approx 528{,}390$ **Note answers for these types of questions in this chapter may not be for the current year.**
   b. What was the population one year before the most current estimate?
      $513{,}000 \cdot 1.03^{-1} \approx 498{,}058$
   c. Express the population *P* as a function of *n*, the number of years from now.
      $P(n) = 513{,}000 \cdot 1.03^n$
   d. What will be the population in 25 years? What assumption do you need to make in order to answer this part?
      $513{,}000 \cdot 1.03^{25} \approx 1{,}074{,}108$
      I assumed that the growth rate will remain constant.
   e. How long will it take until the population is twice what it is today? **About 25 years.** Four times? **About 50 years.** Eight times? **About 75 years.** Does Western Sahara have a relatively high growth rate? Explain. **Western Sahara has an extremely high growth rate. In 50 years, a person's expected lifetime in Western Sahara, the population will increase by a factor or 4. In 75 years, a person's expected lifetime in the USA, the population will increase by a factor or 8. That would mean 8 times as many schools, 8 times as many hospitals, 8 times as many house, roads, crimes, and Western Sahara would need 8 times as much food and water. A population growth rate of 3.0% is huge.**

2. Find a-d above if the population growth you found had been labeled "continuous growth rate". What did you notice about the results in 1 and 2?
   a. What will the population be one year from the most current estimate?
      $513{,}000 \cdot e^{0.03 \cdot 1} \approx 528{,}623$
   b. What was the population one year before the most current estimate?
      $513{,}000 \cdot e^{0.03 \cdot (-1)} \approx 497{,}839$
   c. Express the population *P* as a function of *n*, the number of years from now.
      $P(n) = 513{,}000 \cdot e^{0.03 \cdot n}$
   d. What will be the population in 25 years? What assumption do you need to make in order to answer this part?

$P(25) = 513{,}000 \cdot e^{0.03 \cdot 25} \approx 1{,}086{,}021$

I assumed that the growth rate will remain constant.

3. Does Portugal have a relatively high growth rate (Use CIA World Factbook)? Explain. **Portugal has an annual population growth rate of 0.2%. At this rate, it will take about 345 years for the population of Portugal to double. A population growth rate of 0.2% is relatively small. You might want to look up the growth rate of various countries in the world, guessing the growth rates before you look.**

4. Make up context or situation for which the relationship between x and y is $y = 300 \cdot 1.02^x$. Make sure you make clear the meaning of 300 and 1.02.
It is estimated that there are currently 300 deer in a small county in Rhode Island and that the population is growing at about 2% per year. After x years, there will be $300 \cdot 1.02^x$ deer.

5. Various credit card interest rates are given below for June of 2009.
   a. Approximately how long would it take for debt to double at the national average?

      Using the rule of 70...

      $$11.94 \cdot t = 70$$
      $$t = \frac{70}{11.94}$$
      $$t \approx 5.9$$

      In about 5.9 years, the credit card debt would double.
   b. Approximately how long would it take for debt to double at the "Student" rate?

      Using the rule of 70...

      $$14.45 \cdot t = 70$$
      $$t = \frac{70}{14.45}$$
      $$t \approx 4.8$$

      In about 4.8 years, the credit card debt would double.

Credit Card Rate Report	
Updated: 06-30-2009	
National Average	11.94%
Balance Transfer	9.98%
Low Interest	10.41%
Cash Back	11.20%
Business	11.24%
Reward	12.03%
Instant Approval	12.49%
Airline	13.22%
Bad Credit	14.29%
Student	14.45%

Source: CreditCards.com Home Page, http://www.creditcards.com/, 6/30/09

6. For each description of an exponential function $f(x) = a \cdot b^x$, find a and b.
   a. f(0)=3 and f(1)=12

      f(0)=3 and f(1)=12

      $3 = a \cdot b^0 \Rightarrow a = 3$

      $12 = a \cdot b^1 \Rightarrow 12 = 3 \cdot b \Rightarrow b = 4$

      $f(x) = 3 \cdot 4^x$

   b. f(0)=4 and f(2)=1 (This problem written by the Phillips Exeter Academy Math Department)

      f(0)=4 and f(2)=1

      $4 = a \cdot b^0 \Rightarrow a = 4$

      $1 = a \cdot b^2 \Rightarrow 1 = 4 \cdot b^2 \Rightarrow b = \frac{1}{2}$

      $f(x) = 4 \cdot \frac{1}{2}^x$

7. Fill in the empty boxes by continuing the obvious pattern.
   Verify with your calculator.

10,000	10^4
1,000	10^3
100	10^2
10	10^1
1	$\mathbf{10^0}$
$\frac{1}{10}$	$\mathbf{10^{-1}}$
$\frac{1}{100}$	$\mathbf{10^{-2}}$
$\frac{1}{1000}$	$\mathbf{10^{-3}}$

8. Use what you learned in (6) to evaluate:
   a. 10^{-5} **= 1/100000 = .00001**
   b. 2^{-1} **= 1/2**

c. 2^{-3} **= $(½)^3$ = 1/8**
d. 5^0 **= 1**

Problem set 7-3

1. Use the on-line CIA World Factbook (Search "CIA World Factbook" with Google) to find the current population and annual population growth rate of Latvia.
   a. What will the population be in one year from the most current estimate?
      $2{,}306{,}306 \cdot 0.9929^{1} \approx 2{,}289{,}931$
      (note: b=1+r, 0.9929=1+(-.0071))
   b. What was the population one year before the most current estimate?
      $2{,}306{,}306 \cdot 0.9929^{-1} \approx 2{,}322{,}798$
   c. Express the population P as a function of n, the number of years from now. $P(n) = 2{,}306{,}306 \cdot 0.9929^{n}$
   d. What will be the population in 25 years?
      $2{,}306{,}306 \cdot 0.9929^{25} \approx 1{,}929{,}988$
      I assumed that the growth rate will remain constant.

2. According to the PEW Research Center, the portion of the US Population that identifies themselves as Christian declined about 7% from 2007 to 2014. At this rate, how many years will it take until there the number of Christians is half what it is today? Is it possible to use the rule of 70 in this case? Explain.

   Source: "America's Changing Religious Landscape." Religion and Public Life. PEW Research Center, n.d. Web. 16 June 2015. <http://www.pewforum.org/2015/05/12/americas-changing-religious-landscape/>.

3. The Smithsonian Book of North American Mammals reports that, "Steller sea lion population numbers have declined by more than 90 percent in the last 20 years in most of Alaska and southern California". If the population declined by the same percent each of those 20 years, by what percent did it decline each year? (Assume that the population declined by exactly 90%.)
   90% decline means 10% left.

   $0.10P = P \cdot (1+r)^{20}$

   $0.10 = (1+r)^{20}$ (Raise both side to the $\frac{1}{20}$)

   $1+r = 0.1^{\frac{1}{20}}$

   $r = 0.1^{\frac{1}{20}} - 1$

   $r \approx -0.11$

   If the rate of decline was the same each year,
   it would have been about 11% decline each year.

   Source: The Smithsonian Book of North American Mammals (Natural History). P.199, Vancouver: Univ Of British Columbia Pr, 2003.

4. If you buy a car for $29,873 after one year with typical driving distances it is only worth about $27,314. If the value of the car continues to depreciate at the same rate, what will the car be worth after 5 years?
"Depreciation Infographic: How Fast Does My New Car Lose Value?" *edmunds.com*. N.p., n.d. Web. 31 May 2016. <http://www.edmunds.com/car-buying/ how-fast-does-my-new-car-lose-value-infographic.html>.

$$\textit{The new value} \cdot \textit{the (decay) factor} = \textit{the value in one year}$$

$$\$29{,}873 \cdot b = \$27{,}314$$

$$b = \frac{27{,}314}{29{,}873}$$

In general, $y = a \cdot b^x$.

In 5 *years*, *the car will be worth*…

$$y = \$29{,}873 \cdot \left(\frac{27{,}314}{29{,}873}\right)^5 \approx \$19{,}090.23$$

5. For the description of an exponential function $f(x) = a \cdot b^x$, find a and b.
f(3)=2 and f(5)=32
(This problem is adapted from a problem written by Phillips Exeter Academy Math Department)

f(3)=2 and f(5)=32

$2 = a \cdot b^3 \Rightarrow a = 2/b^3$

$32 = a \cdot b^5 \Rightarrow a = 32/b^5$

$$\frac{2}{b^3} = \frac{32}{b^5}$$

$b^2 = 16 \Rightarrow b = 4$

Use b=4 to find a. $a = 2/b^3 \Rightarrow a = 2/4^3 = 0.03125$

$f(x) = 0.03125 \cdot 4^x$

6. Fungus is growing exponentially in a Petri dish in a circular pattern according to the function $f(x) = a \cdot b^x$ where *a* is the initial area, *b* is the growth factor and f(x) is the area after x hours. Two hours after the start of the experiment the area of the fungus was 5mm^2. After 4 hours the area was 17 mm^2.

a. What was the growth factor?

f(2)=5 and f(4)=17

$5 = a \cdot b^2 \Rightarrow a = 5/b^2$

$17 = a \cdot b^4 \Rightarrow a = 17/b^4$

$$\frac{5}{b^2} = \frac{17}{b^4}$$

$$b^2 = \frac{17}{5} \Rightarrow b = \sqrt{\frac{17}{5}} = \left(\frac{17}{5}\right)^{\frac{1}{2}}$$

b. What was the initial area?

Use $b=\sqrt{\frac{17}{5}}$ to find a.

$$a = \frac{5}{b^2} \Rightarrow a = \frac{5}{\left(\left(\frac{17}{5}\right)^{\frac{1}{2}}\right)^2} = \frac{25}{17}$$

$\approx 1.47\,mm^2$

c. Give a function for the area in terms of time since the start of the experiment.

$$f(x) = \left(\frac{25}{17}\right) \cdot \sqrt{\frac{17}{5}}^{\,x} \quad \textit{or} \quad f(x) = \left(\frac{25}{17}\right) \cdot \left(\frac{17}{5}\right)^{\frac{x}{2}}$$

What was the diameter 7 hours after the start of the experiment?

$$f(7) = \left(\frac{25}{17}\right) \cdot \sqrt{\frac{17}{5}}^{\,7} = \left(\frac{25}{17}\right) \cdot \left(\frac{17}{5}\right)^{\frac{7}{2}} \approx 106.6$$

$\approx 106.6\ mm^2$

d.

7. Show how you can evaluate the following without a calculator.

a. $100^{\frac{1}{2}}$ $100^{\frac{1}{2}} = \sqrt{100} = 10$

b. $16^{\frac{3}{2}}$ $16^{\frac{3}{2}} = \left(\sqrt[2]{16}\right)^3 = 4^3 = 64$

c. 7^{-1} $7^{-1} = \frac{1}{7}$

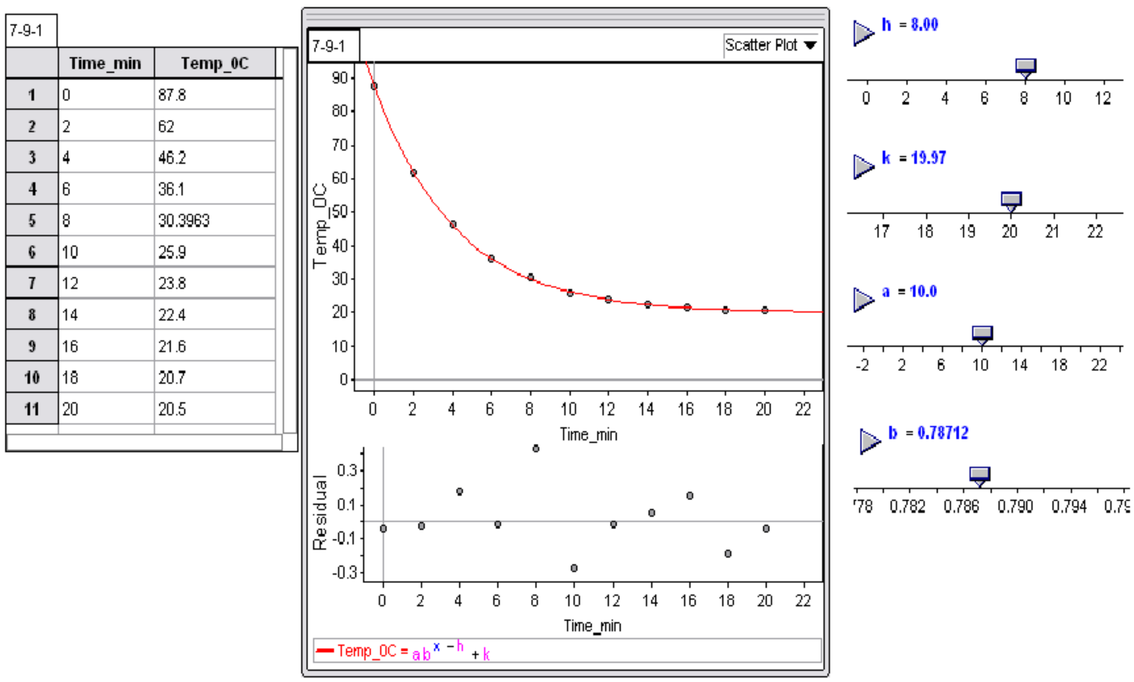
d. $27^{-\frac{4}{3}}$ $27^{-\frac{4}{3}} = \left(\left(\sqrt[3]{27}\right)^4\right)^{-1} = \left((3)^4\right)^{-1} = (81)^{-1} = \frac{1}{81}$

Problem set 7-4

1. A cup of hot coffee was 87.8 ^{0}C when it was poured and the temperature was measured every 2 minutes for 20 minutes as shown below.

Time (minutes)	Temperature (^{0}C)
0	87.8
2	62.0
4	46.2
6	36.1
8	30.1
10	25.9
12	23.8
14	22.4
16	21.6
18	20.7
20	20.5

a. Plot the data on a scatterplot. Fit an exponential model to the data and draw it on the graph with the scatterplot. Express the Temperature *T* as a function of the time *t*.



Meaning of the Decay Factor

$T = 10 \cdot 0.787^{t-8} + 19.97$

b. Write one sentence each that explains the meaning the y-intercept, and the horizontal asymptote within the context of the problem.

c. According to your model, what was the temperature after 11 minutes?

$10 \cdot 0.787^{11-8} + 19.97 \approx 24.84\ ^0$C

d. According to your model, when was the temperature 21 C^0?

$21 = 10 \cdot 0.787^{t-8} + 19.97$

Use CAS (a Computer Algebra System) to solve for t

$t \approx 17.5$ minutes

2. Mauna Loa CO^2 data revisited
💻 The Fathom file in Course Materials for section 7-9 contains CO^2 concentrations in ppm (parts per million) since 1964.
   a. Fit an exponential model to the data, but don't worry about the seasonal fluctuations.
   b. According to your model, what will the CO^2 concentration be in 50 years, about when you will retire?
   c. What is the growth rate of CO^2 concentration? Is that a high growth rate? Explain.

3. 💻 The CIA World Factbook reported that in 2007 the US consumed 3,892 TWh of electricity. TWh stands for Terra Watt Hours. Terra means 10^{12}.
   a. Open up the file "Wind Energy Projections" and fit an appropriate model using Year as the independent variable and Wind Output as the dependent variable. **One possible model is** $y = 380 \cdot 1.2^{x-2012}$
   b. According to your model, when will the US produce half of its electricity from Wind Power? **About 2021**

   Sources:
   US CIA, US Country Information, https://www.cia.gov/library/publications/the-world-factbook/geos/US.html, 6/13/09
   Energy Watch Group, New Report Shows Wind Power Experiencing Exponential Growth, http://www.energywatchgroup.org/fileadmin/global/pdf/2009-01-09_EWG_Press_Wind_Report.pdf , 6/13/09

Problem set 7-5

1. Write each of the following numbers as a power of 10. You should be able to do this without your calculator. (PEA)

   a. 100 b. 100000 c. $\frac{1}{1000}$ d. $\sqrt{10}$ e. $100\sqrt{10}$ f. $\frac{1}{\sqrt[3]{100}}$

   a. 10^2 **b.** 10^5 **c.** 10^{-3} **d.** $10^{\frac{1}{2}}$ **e.** $10^{2\frac{1}{2}}$ **f.** $10^{-\frac{2}{3}}$

2. Now use your calculator. Press LOG, followed by each of the following. Be careful with your parenthesis. (PEA)

   a. 100 b. 100000 c. $\frac{1}{1000}$ d. $\sqrt{10}$ e. $100\sqrt{10}$ f. $\frac{1}{\sqrt[3]{100}}$

   a. 2 **b. 5** **c. -3** **d. 0.5** **e. 2.5** **f. -0.666…**

3. Describe in you own words what LOG "does".

4. If $f(x) = 10^x$ then find:
   a. $f^{-1}(100)$ **2**
   b. $f^{-1}(1{,}000)$ **3**
   c. $f^{-1}(100{,}000)$ **5**

5. Once you know that $\log_{10} 100 = 2$ is just another way of writing $10^2 = 100$ you can rewrite exponential equations as equivalent logarithmic equations and vice versa.
   a. Write an equation equivalent to $2^7 = 128$ using logs. $\log_2 128 = 7$
   b. Write an equation equivalent to $\log_6 216 = 3$ using exponents. $6^3 = 216$

6. Solve the following equations. First give the exact value and then an approximation to 2 decimal places.
   a. $2^x = 9$

      $2^x = 9$

      $x = \log_2 9$

      $x \approx 3.17$
   b. $e^x = 9$

      $e^x = 9$

      $x = \ln 9$

      $x \approx 2.20$

7. Solve $\log_x 100 = 2$.

$$\log_x 100 = 2$$
$$x^2 = 100$$
$$x = 10$$

8. At the beginning of 2004, the population of the USA was about 293,045,000 and the population growth rate was about 0.92%. If the growth rate stays constant, when will the population of the USA reach 1 billion people? First give a) the exact answer and then b) an approximation.

$$p = 293{,}045{,}000 \cdot 1.0092^x$$
$$1{,}000{,}000{,}000 = 293{,}045{,}000 \cdot 1.0092^x$$
$$\frac{1{,}000{,}000{,}000}{293{,}045{,}000} = 1.0092^x$$
$$x = \log_{1.0092}\left(\frac{1{,}000{,}000{,}000}{293{,}045{,}000}\right)$$
$$x \approx 134.0 \text{ years}$$

9. Answer 8) above if the population growth was labeled “continuous growth rate”.

$$p = 293{,}045{,}000 \cdot e^{0.0092 \cdot x}$$
$$1{,}000{,}000{,}000 = 293{,}045{,}000 \cdot e^{0.0092 \cdot x}$$
$$\frac{1{,}000{,}000{,}000}{293{,}045{,}000} = e^{0.0092 \cdot x}$$
$$\ln\left(\frac{1{,}000{,}000{,}000}{293{,}045{,}000}\right) = 0.0092 \cdot x$$
$$x = \frac{\ln\left(\frac{1{,}000{,}000{,}000}{293{,}045{,}000}\right)}{0.0092}$$
$$x \approx 133.4 \text{ years}$$

10. Consider your ancestry consisting of 2 parents, 4 grandparents, 8 great grandparents, and so on. How many generations do you need to go back until you have at least 1,000,000 great-great-great… grandparents? That is, don't count the parents, grandparents, etc. along the way; just count the number of ancestors that many generations back. Use logarithms to find the exact answer, then give an approximation to the nearest tenth of a year. Test your instincts – write down a guess before you make any calculations.

$1{,}000{,}000 = 1 \cdot 2^x$

$x = \log_2 1{,}000{,}000$

$x = \dfrac{\log 1{,}000{,}000}{\log 2} \approx 19.9$ generations

You need to go back 20 generations.

11. In June of 2009, Vice President Joe Biden said the following on "Meet the Press": "We have health care going up… 50 percent in the last six years.

a. If health care costs went up by the same percent each of those six years, by what percent did it go up in one year?

$1.5P = P \cdot (1+r)^6$

$1.5 = (1+r)^6$ (Raise both side to the $\frac{1}{6}$)

$1 + r = 1.5^{\frac{1}{6}}$

$r = 1.5^{\frac{1}{6}} - 1$

$r \approx 0.07$

If the growth rate was the same each year,
it would have been increase each year by about 7%.

b. Given your answer to (a), how long would it take for health care costs to double? Give your answer to the nearest tenth of a year.

Using the "rule of 70" for doubling, we know the answer is about 10 years.

More accurately (to the nearest tenth)...

$2P = P \cdot (1+r)^t$ Dividing both sides by P

$2 = (1 + \left(1.5^{\frac{1}{6}} - 1\right))^t$ Substituting for r

$t = \log_{1+\left(1.5^{\frac{1}{6}}-1\right)} 2$

$t = \log_{1.5^{\frac{1}{6}}} 2$

$t \approx 10.3$

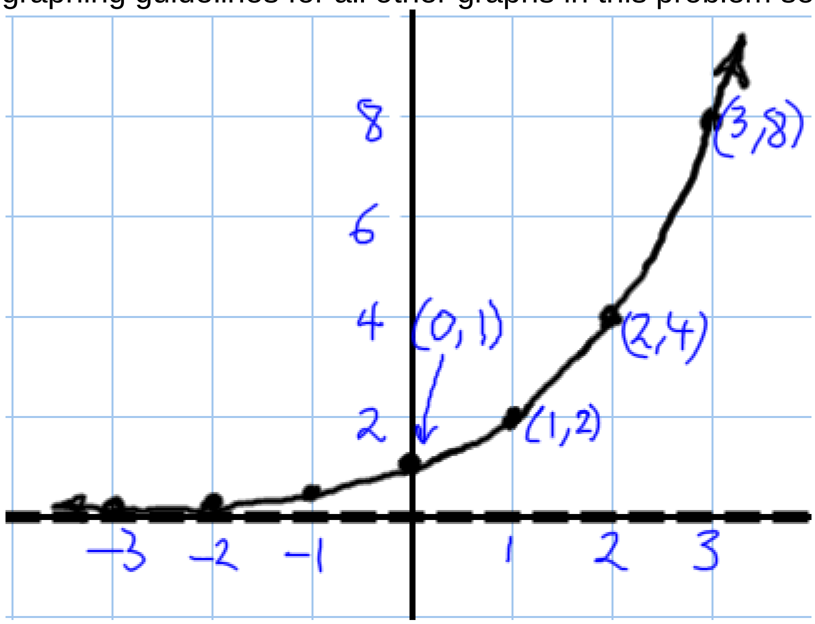
If the growth rate was the same each year,

it would take about 10.3 years for costs to double.

Source: Vice President Joseph Biden, 'Meet the Press' transcript for June 14, 2009, http://www.msnbc.msn.com/id/31343018/ns/meet_the_press_online_at_msnbc/page/3/, 6/14/2009

Problem set 7-6

1. Consider $f(x) = 2^x$.
   a. Graph $y= f(x)$. Label two anchor points and include x- and y-intercepts if they exist. Indicate all asymptotes with a dashed line. Use these graphing guidelines for all other graphs in this problem set.



   b. What is the domain of f?
      All real numbers
   c. What is the range of f?
      y: y>0

2. If $f(x)=2^x$
   a. Find $f^{-1}(x)$.

$f(x)=2^x$

$y = 2^x$

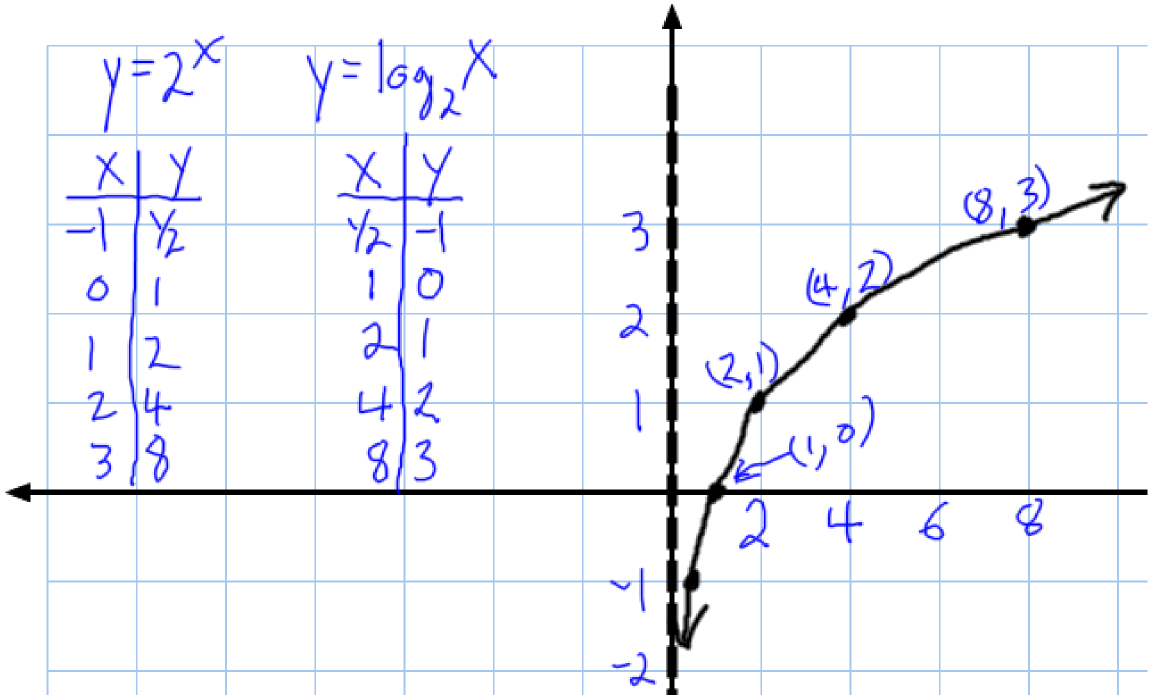
Switch x and y to find the inverse

$x = 2^y$

$y = \log_2 x$

$f^{-1}(x) = \log_2 x$

b. Graph $y = f^{-1}(x)$.



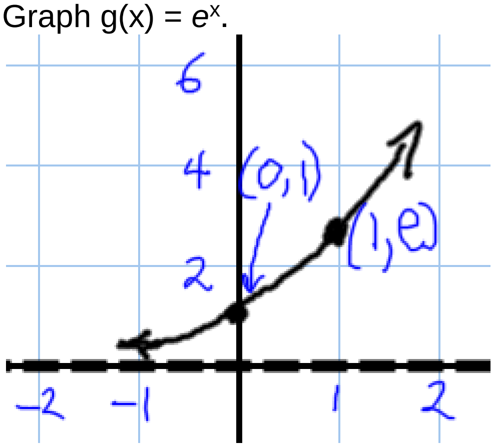
c. What is the domain of f^{-1}?

x: x>0

d. What is the range of f^{-1}?

All real numbers

3. Graph $g(x) = e^x$.



4. If g(x)= e^x

a. Find $g^{-1}(x)$.

$g(x)=e^x$

$y = e^x$

Switch x and y to find the inverse

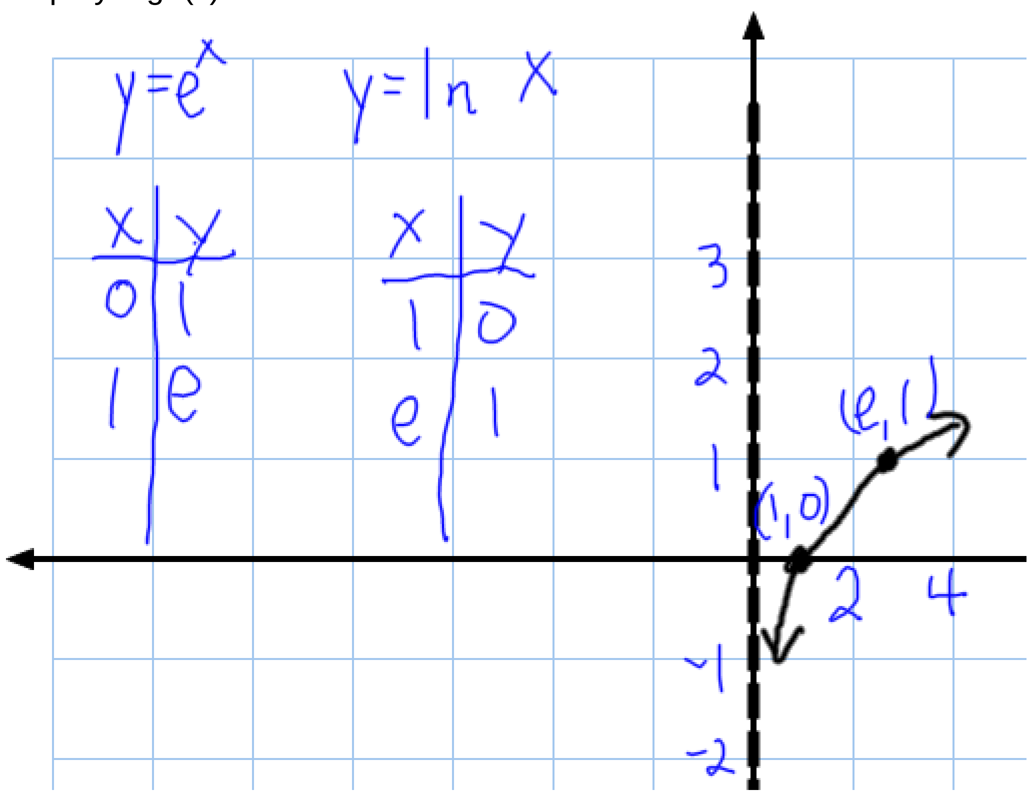
$x = e^y$

$y = \log_e x$ (log$_e$ is typically only used while learning about natural logs. Use ln.

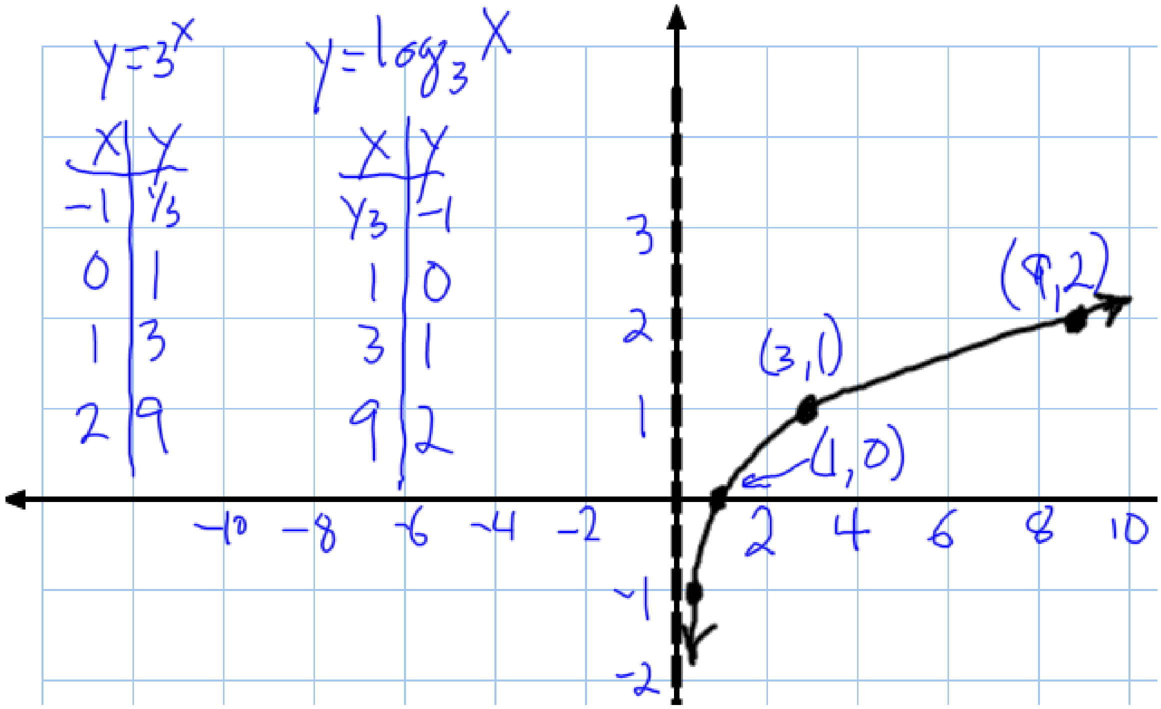
$y = \ln x$

$g^{-1}(x) = \ln x$

b. Graph $y = g^{-1}(x)$.



5. Graph $y = \log_3 x$ for $-10 \le x \le 10$.



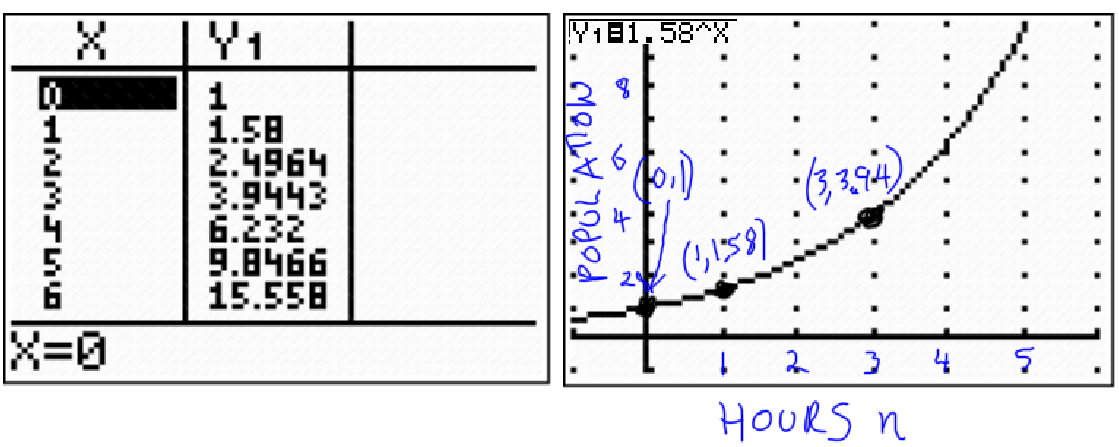
6. Yeast grows asexually, via a process called budding. As a result, one yeast cell, under the right conditions, can grow into a colony of yeast cells. Yeast grows exponentially at the rate of about 58% per hour until it reaches a density of about 4×10^7 cells/ml.

Source: Introduction to Yeast, Yeast resources at Duke University, http://www.dbsr.duke.edu/yeast/Info%20and%20Protocols/Growth.htm, 3/5/05

a. Express the population *P* of yeast as a function of *n*, the number of hours since the yeast was started to bud.

$P(n) = 1(1.58)^n$

b. Graph the function for hours 0 through 5, labeling anchor points for each hour.



c. Express the number of hours *n*, as a function of the population *P*. Graph this function including all of the same anchor points (switching the x- and y-coordinates).

$P(n)=1.58^x$

$y = 1.58^x$

Switch x and y to find the inverse

$x = 1.58^y$

$y = \log_{1.58}x$ or $n(P) = \log_{1.58}P$

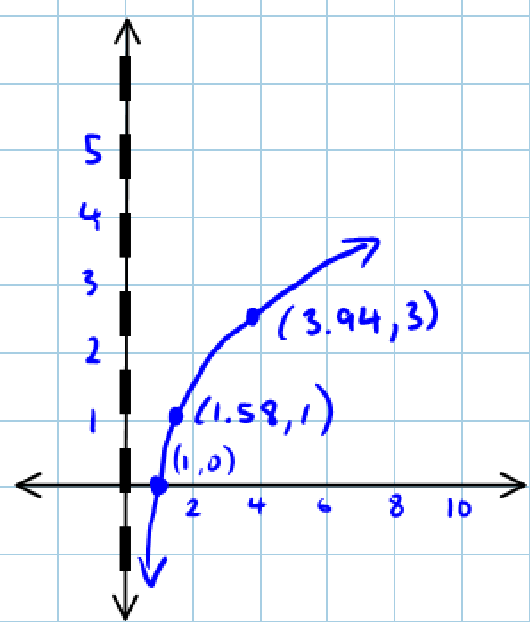
To graph $y = \log_{1.58}x$, switch the x- and y-coordinates of $y = 1.58^x$.

$y = 1.58^x$

x	y
0	1
1	1.58
3	3.94

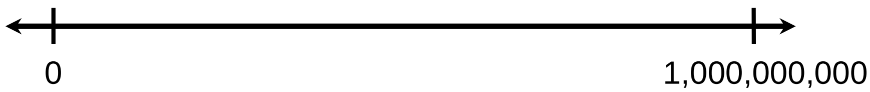
$y = \log_{1.58}x$

x	y
1	0
1.58	1
3.94	3



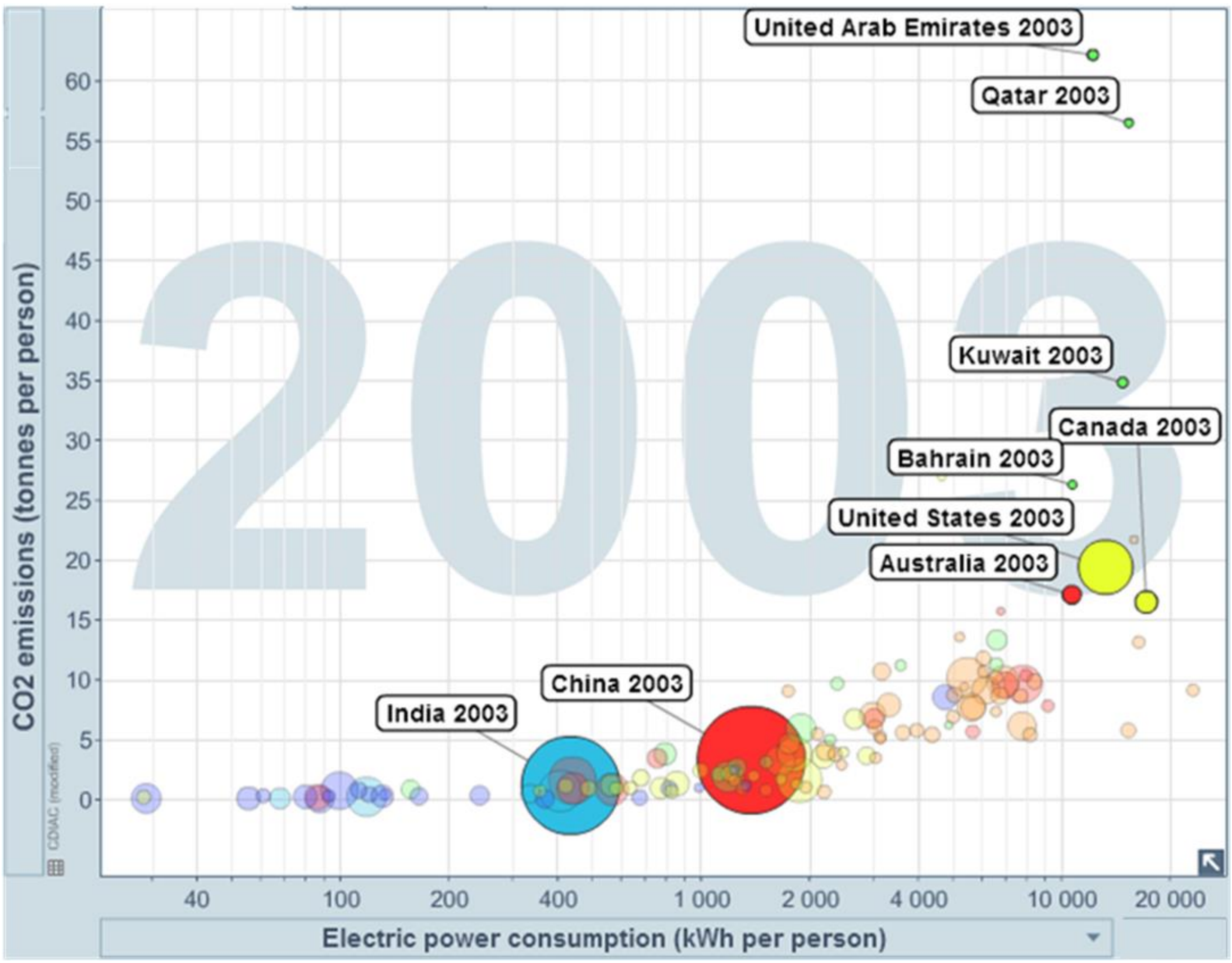
Problem set 7-7

1. Consider the following number line. Put a point on the number line that represents the position of 1,000,000. Explain exactly where you put the point and why. (Jeanloz – U. Cal. Berkeley)



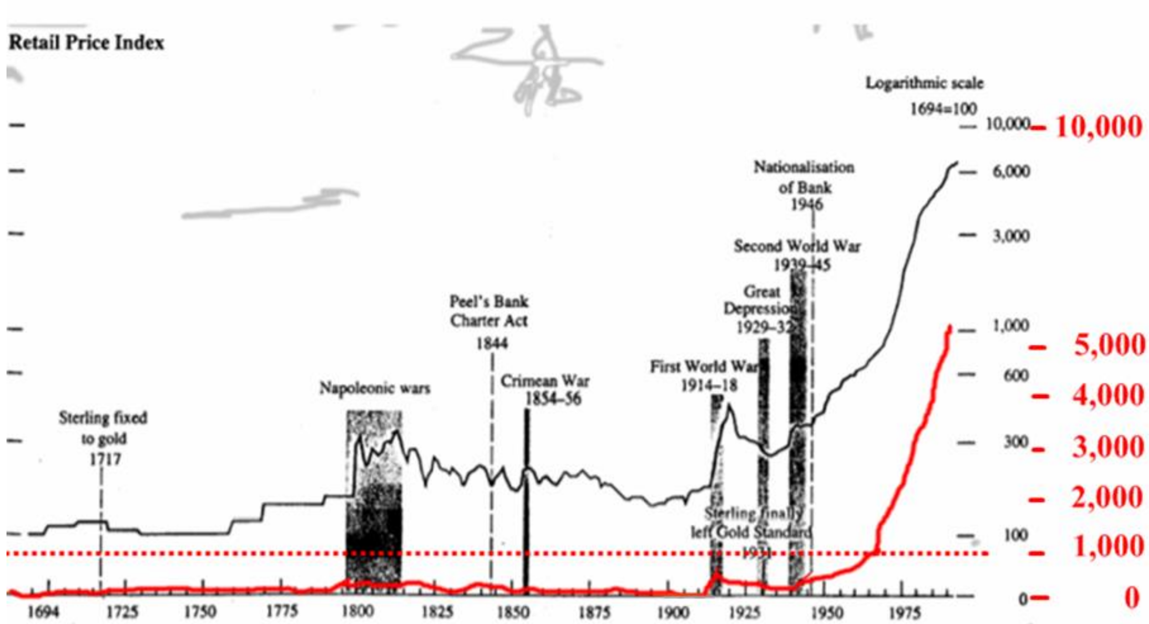
2. If you were to graph the following variable, would you use a linear or logarithmic scale and why?
   a. year from 1990 – 2010
   linear; the values of the variable “year from 1990-2010” do not span more than two orders of magnitude
   b. year from 1500 – 2000
   linear; the values of the variable “year from 1500-2000” do not span more than two orders of magnitude
   c. population of a country (all countries in the world included in data set)
   log; the values of the variable “population of a country” do span more than two orders of magnitude
   d. fuel efficiency of a passenger vehicle in miles per gallon
   linear; the values of the variable “fuel efficiency of a passenger vehicle in miles per gallon” do not span more than two orders of magnitude
   e. how much a person was in debt
   Although debt would most likely span more than two orders of magnitude, some of the people would probably not have any debt so you could not use a log scale. Zero is not on a log scale.
   f. age at death of a human, rounded to the nearest year
   linear; the values of the variable “age at death” do not span more than two orders of magnitude
   g. number of total heartbeats of a human in a lifetime
   log; the values of the variable “number of total heartbeats of a human in a lifetime” do span more than two orders of magnitude

3. Consider the Gapminder graph below.
   a. What type of scale is used on the x-axis and why?
   A logarithmic scale is used on the x-axis because the values of Electric Power Consumption Per Person span more than two orders of magnitude.
   b. What type of scale is used on the y-axis and why?
   A linear scale is used on the y-axis because the values of CO2 Emissions Per Person do not span more than two orders of magnitude.



Source: Gapminder Foundation, CO2 Emissions vs. Electricity Consumption, http://graphs.gapminder.org/world , 7/1/09

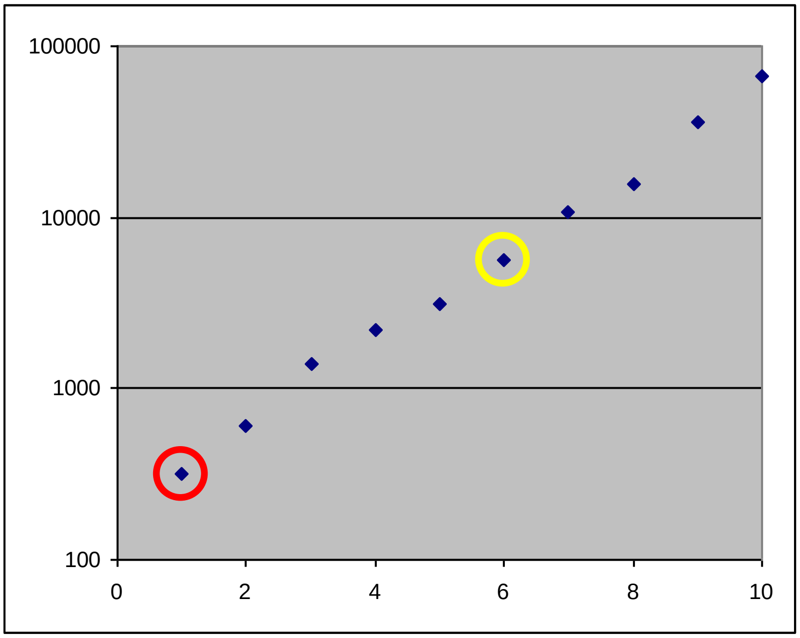
4. Consider the graph below: A retail price index for UK 1694-1994
   Explain in a complete sentence, why Holter used a logarithmic Scale.
   The Retail Price Index spans more than 2 orders of magnitude. If Holter would have used an arithmetic scale, it would not have been possible to distinguish between any of the y-coordinates of the points between 1694 and about 1925 where the RPI was below 60 (about 2 orders of magnitude below 6,000 which was the largest value). The reader would not have noticed, for example, that the Retail Price Index went up during the Napoleonic Wars.



Source: A historical perspective on monetary statistics in Norway, Jon Holter, http://www.norges-bank.no/stat/historiske_data/en/hms/c2.pdf ,3/4/05.

5. Note the graph above which has zero on the logarithmic scale. Is it mathematically correct to have zero be on a logarithmic scale?
   No. A logarithmic scale is a scale that has equally spaced tick marks and a geometric sequence associated with consecutive tick marks. The only geometric sequence that includes 0 is a, 0, 0, 0, …

6. Estimate the Y-coordinates of the points circled on the scatter plot.
   a. Red
      The point appears to be halfway between 100 and 1,000. $100=10^2$ and $1000=10^3$ so on a logarithmic scale the point half way between 10^2 and 10^3 is $10^{2.5} \approx 316$.
   b. Yellow **$10^{3.75} \approx 5623$**



7. A review question - A CD (Certificate of Deposit) pays 6% compounded continuously.
   a. Exactly how long will it take for the money in the account to double?

$$A = P \cdot \left(e^r\right)^t$$

$$2P = P \cdot \left(e^{.06}\right)^t$$

$$2 = e^{.06\,t}$$

$$\ln 2 = .06\,t$$

$$t = \frac{\ln 2}{.06}$$

   b. Approximately how long will it take for the money in the account to double?
      $x \approx 11.6$ years

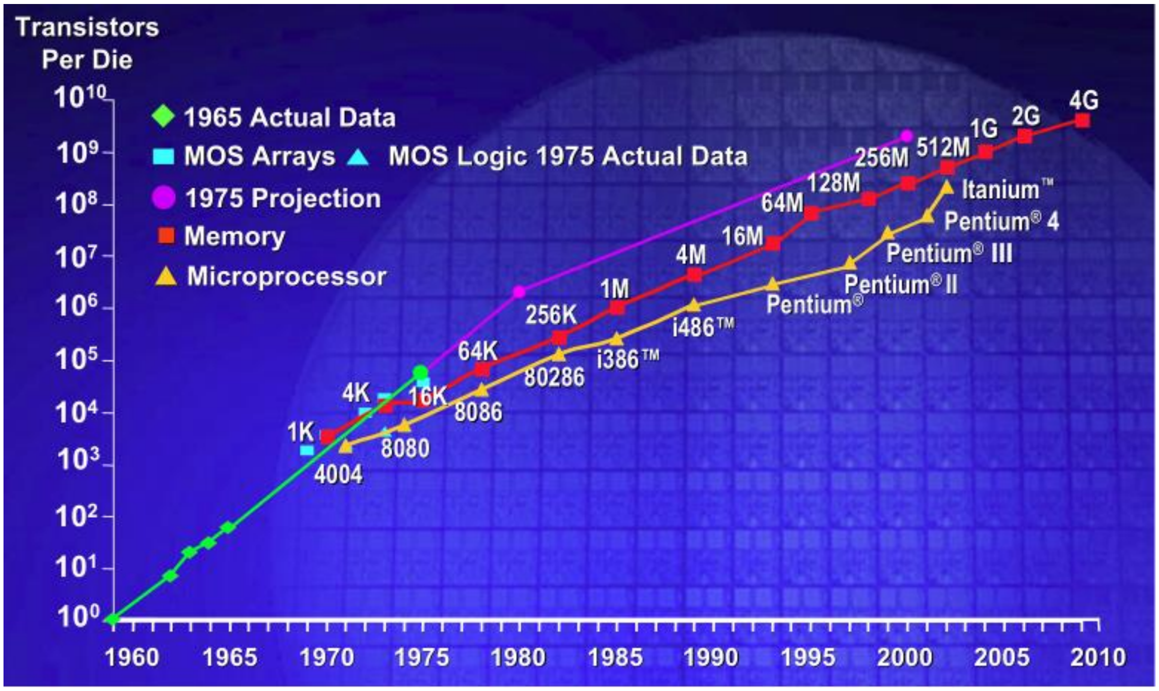
   c. What is the EAY (effective annual yield) for this CD?

$$EAY = e^{.06} - 1$$

$$EAY \approx 0.0618 \text{ or } 6.18\%$$

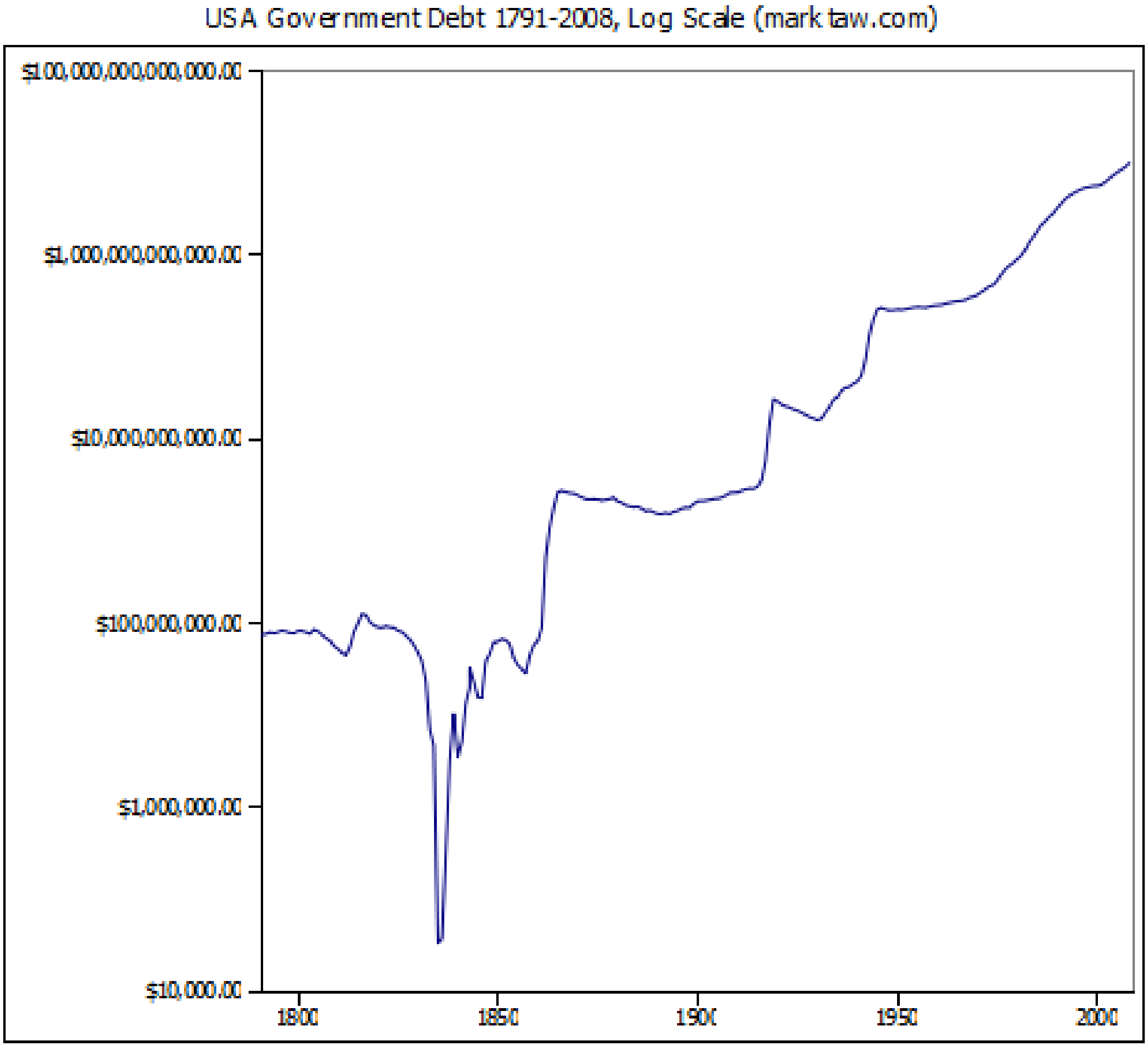
Problem set 7-8

1. CMG's article about Moore's Law gives the following graph. "Transistors Per Die" is a measure of how much memory a computer has. Based on the graph (use the red scatter plot labeled "Memory"), what can you conclude about computer memory? Justify your answer. Note, you answer is Moore's Law.
The graph has 1) a linear scale on the x-axis, 2) a logarithmic scale on the y-axis, and 3) the scatterplot is very linear. Therefore computer memory has been growing exponentially since 1970.



Source: Computer Measurement Group, Inc., "Moore's Law: More or Less?", http://www.cmg.org/measureit/issues/mit41/m_41_2.html, 7/1/09

2. Based on the graph below, what can you conclude about US Inflation Adjusted National Debt since 1900? Justify your answer.



Source: Mark Wieczorek, USA Inflation Adjusted Government Debt 1791-2008, http://www.marktaw.com/culture_and_media/politics/USA_debt_2008.html, 1/7/09

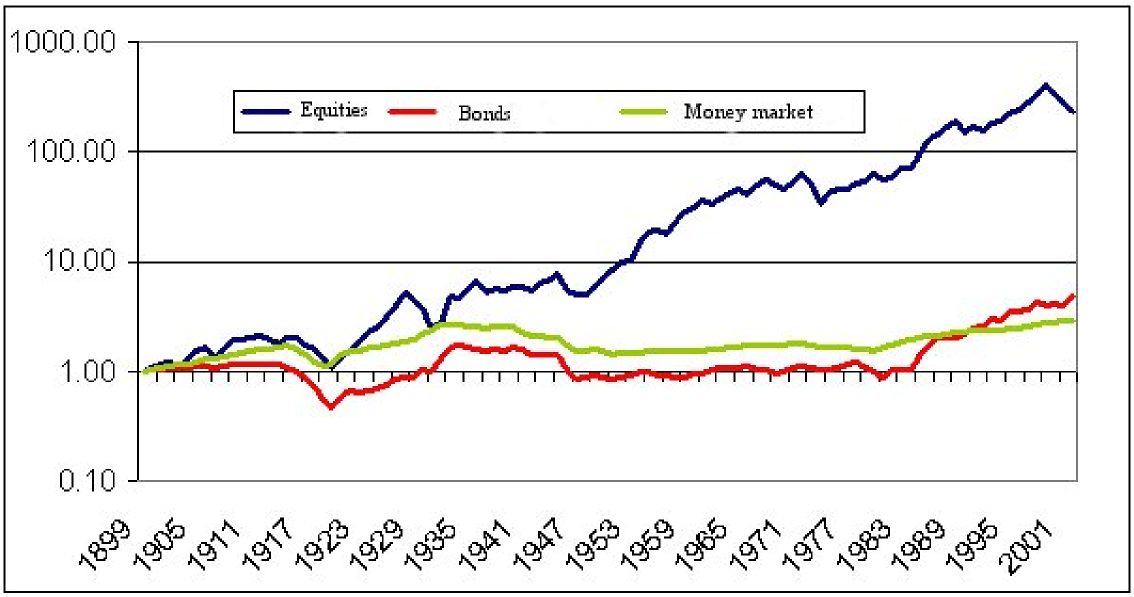
The graph has 1) a linear scale on the x-axis, 2) a logarithmic scale on the y-axis, and 3) the scatterplot is fairly linear (no overall concavity). Therefore US Debt has been growing exponentially since 1900.

3. Consider the chart below of Cumulative Real Returns vs. year. What can you conclude about the “Equities” market from 1935 to 1995? Justify your answer.
The graph has 1) a linear scale on the x-axis, 2) a logarithmic scale on the y-axis, and 3) the scatterplot is fairly linear (no overall concavity). Therefore Equities have been growing exponentially since 1899.

Excess return on shares

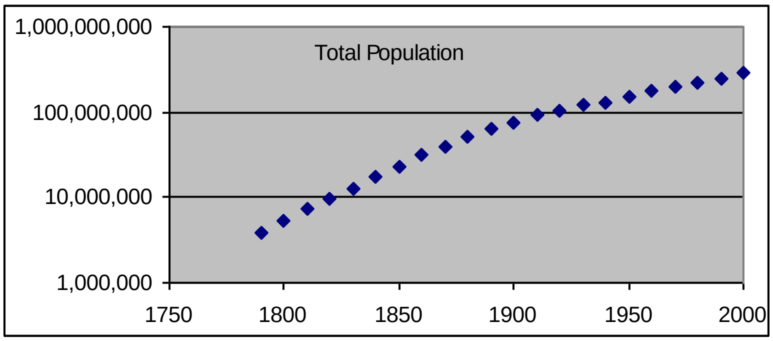
Chart 2 shows the real return from 1900 for three investors who at the beginning of 1900 invested their capital in shares, bonds or the US money market. The money market investments are in short-term US government paper and may thus be said to be the least risky alternative for an investor in the short-term.

Chart 2: Cumulative real return on global capital investments from 1900 to 2002 (logarithmic scale)

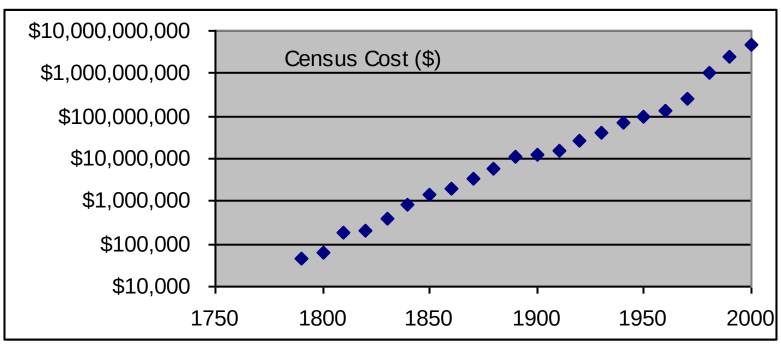


Source: Norges Bank, 103 years in the capital markets, http://www.norges-bank.no/english/petroleum_fund/articles/103_years_2003/ , 3/4/05.

4. 💻 Open up the Excel Spreadsheet labeled "US Census Data (Excel)". Change the y-axis to a logarithmic scale for each of the three graphs (a, b, and c). For each graph, comment on whether an exponential model is an appropriate model for the relationship between the independent and dependent variable. Write your answer in the context of the Census and be sure to include a justification for you answer.
   a. Total Population vs. Year
      The graph has 1) a linear scale on the x-axis, 2) a logarithmic scale on the y-axis, and 3) the scatterplot is not linear (concave down). Therefore the total population did not grow exponentially from 1790 to 2000.

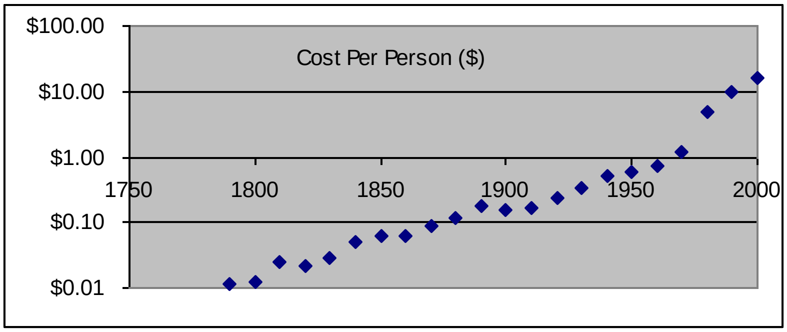


   b. Census Cost vs. Year
      The graph has 1) a linear scale on the x-axis, 2) a logarithmic scale on the y-axis, and 3) the scatterplot is fairly linear (no overall concavity) <u>until about 1970</u>. Therefore the Census Cost grew exponentially from 1790 to 1970.



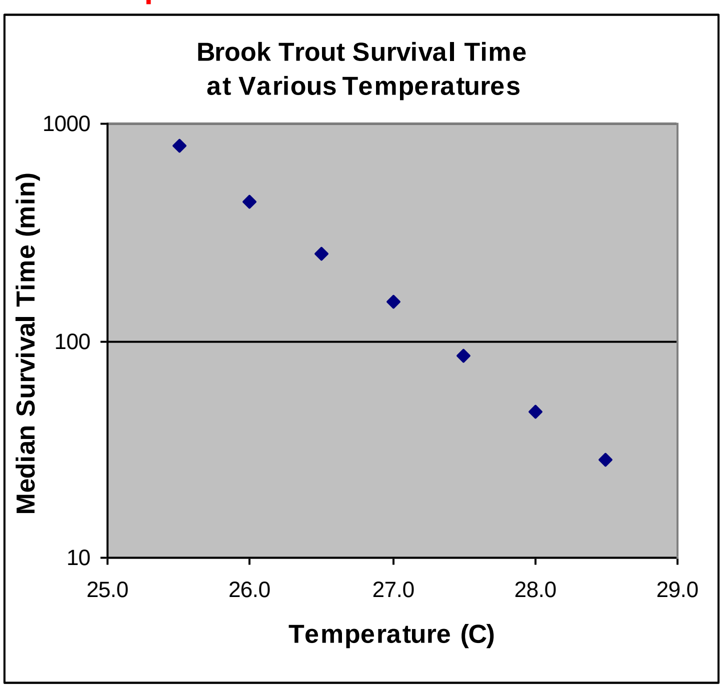
c. Cost per Person vs. Year

The graph has 1) a linear scale on the x-axis, 2) a logarithmic scale on the y-axis, and 3) the scatterplot is fairly linear (no overall concavity) <u>until about 1970</u>. Therefore the Cost per person grew exponentially from 1790 to 1970.



5. 💻 Open up the Excel Spreadsheet labeled "Brook Trout Survival Time (Excel)" and change the y-axis to a logarithmic scale. Comment on whether an exponential model is an appropriate model for the relationship between Mean Survival Time and Temperature. Include a justification.

The graph has 1) a linear scale on the x-axis, 2) a logarithmic scale on the y-axis, and 3) the scatterplot is fairly linear (no overall concavity). Therefore the Mean Survival time of trout decreases exponentially as water temperature increases.



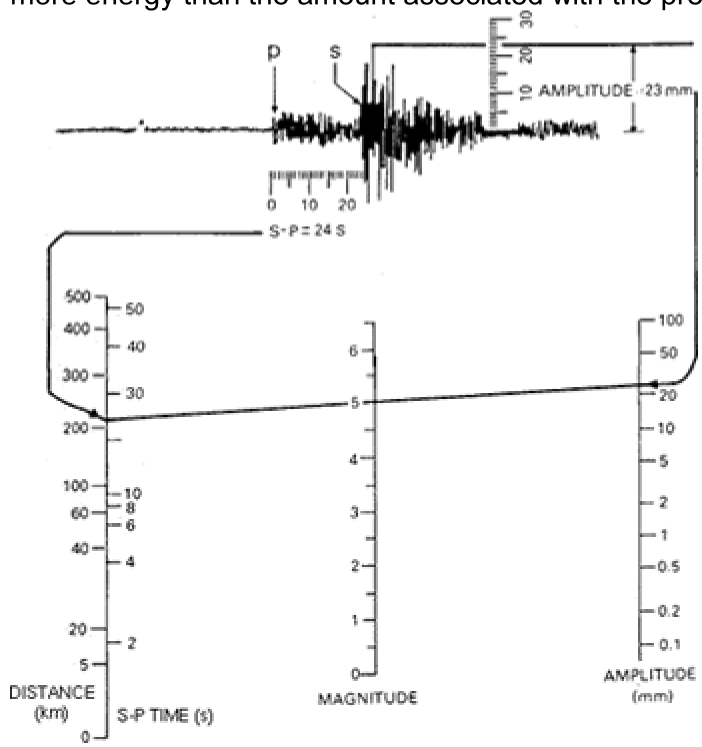
Problem set 7-9

1. Read http://staff.jccc.net/pdecell/chemistry/phscale.html for a brief explanation of the meaning of pH.
   a. Which of the soaps below, relatively speaking, is most acidic? **Dial**
      Which is most basic? **Lever 2000 and Palmolive**
   b. Which is more basic, Palmolive or Ivory? How many times more basic?
      Palmolive is 10 times as basic as Ivory.
   c. Which is more basic, Dial or Dove? How many times more basic?
      We know that Dove is more basic because it has a higher pH, but how much higher?
      The answer: $10^{|\text{Difference of the pHs}|}$.
      In this case: $10^{|7.0-9.5|} = 10^{2.5} \approx 316.2$.
      So, Dove is about 316.2 times as basic as Dial.

   d. If a soap is 300 times as basic as Dial, what is its pH? Give the exact answer and then the answer rounded to the nearest two decimal places.
      $10^{|\text{Difference of the pHs}|} = 300$
      In this case: $10^{|x-7.0|} = 300$.
      $\log 300 = x - 7$
      $x = \log 300 + 7$
      $x \approx 9.48$
      So, the pH of the soap that is 300 times as basic as Dial is about 9.48.

Soap	**pH**
Camay	9.5
Zest	9.5
Dial	7.0
Dove	9.5
Irish Spring	9.5
Ivory	9.0
Lever 2000	10.0
Palmolive	10.0
Source: http://waltonfeed.com/old/soap/soaplit.html	

2. The following is a direct quote and image from the following source:
"Earthquake Glossary - Richter scale." *Earthquake Hazards Program*. USGS, n.d. Web. 15 Apr. 2016. <http://earthquake.usgs.gov/learn/glossary/?term=Richter%20scale>.
"The Richter magnitude scale was developed in 1935 by Charles F. Richter of the California Institute of Technology as a mathematical device to compare the size of earthquakes. The magnitude of an earthquake is determined from the logarithm of the amplitude of waves recorded by seismographs. Adjustments are included for the variation in the distance between the various seismographs and the epicenter of the earthquakes. On the Richter Scale, magnitude is expressed in whole numbers and decimal fractions. For example, a magnitude 5.3 might be computed for a moderate earthquake, and a strong earthquake might be rated as magnitude 6.3. Because of the logarithmic basis of the scale, each whole number increase in magnitude represents a tenfold increase in measured amplitude; as an estimate of energy, each whole number step in the magnitude scale corresponds to the release of about 31 times more energy than the amount associated with the preceding whole number value".



The "Nomograph" above shows that an earthquake that is a little more than 200 km from the seismograph and a wave amplitude of 23mm on the seismograph, is associated with Richter 5.0.

a. In 1967 an earthquake of 6.6 Richter magnitude in Caracas, Venezuela took 240 lives and caused more than $50 million worth of property damage. In 1964 an earthquake of 7.4 Richter magnitude did serious damage in Niigata, Japan, in 1964. How many times greater was the Niigata quake in terms of wave amplitude?

$\frac{10^{7.4}}{10^{6.6}} = 10^{7.4 - 6.6} = 10^{0.8} \approx 6.3$ **The Niigata quake had about 6.3 times the wave amplitude of the Caracas quake.**

b. If there was an earthquake 5.3 Richter and an aftershock that had a wave amplitude that was half of the earthquake, what was the Richter value for the aftershock?

The earthquake had twice the wave amplitude as the aftershock.

$10^{|\text{Difference of the Richter values}|} = 2$

In this case: $10^{|5.3 - x|} = 2$.

$\log 2 = 5.3 - x$

$x = 5.3 - \log 2$

$x \approx 4.6$

So, the aftershock had a Richter value of about 4.6.

3. Take a close look at the Nomograph published by the USGS in the previous problem. There is a mathematical error on it. Find it and describe what the problem is.

4. You might want to actually do the following to help visualize the problem. Take a normal piece of printer paper or photo copy paper and rip in half and stack the two halves on top of each other. Call that "1". Now take the two pieces and rip them in half and stack all the pieces on top of each other to make four pieces. Call that "2". Assume that you were strong enough to keep ripping the stack no matter how thick. How tall will the stack of paper be when you get to "50"? Give the answer both as the number of pieces of paper and a meaningful unit of length.

Project

Read Chapter Two, "Exponential Amplification", from K.C. Cole's book **The Universe and the Teacup: The Mathematics of Truth and Beauty**. Write a self-contained, one page (double-spaced) piece in Cole's style that could be used in this chapter. The reader should have an "aha" experience as a result of reading your piece that brings them to a deeper understanding of exponential growth or decay. Your audience is the general public, not your math teacher or fellow math students.